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Master＇s Thesis
석사 학위논문

# Quadrotor Actuator Fault Detection Considering the Aerodynamic Effect of Propellers 

## Hyukjin Lee（이 혁 진 李 赫 眞）

Department of Information and Communication Engineering
정보통신융합전공

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Advisor: Professor Yongsoon Eun<br>Co-advisor: Professor Kyoung-Dae Kim<br>by<br>Hyukjin Lee<br>Department of Information and Communication Engineering<br>DGIST

A thesis submitted to the faculty of DGIST in partial fulfillment of the requirements for the degree of Master of Science in the Department of Information and Communication Engineering. The study was conducted in accordance with Code of Research Ethics ${ }^{1}$
11. 17. 2020

Approved by
Professor Yongsoon Eun (Signature)
Professor Kyoung-Dae Kim (Signature)

[^0]
# Quadrotor Actuator Fault Detection Considering the Aerodynamic Effect of Propellers 

Hyukjin Lee

Accepted in partial fulfillment of the requirements for the degree of Master of Science.
11. 17. 2020
$\begin{array}{ll}\text { Head of Committee } & \frac{\text { 은 용 순 }}{\text { Prof. }}(\text { 인 }) \\ \text { Yongsoon Eun }\end{array}$
$\frac{\text { Committee Member } \quad \text { 김 경 대 (인) }}{\text { Prof. Kyoung-Dae Kim }}$
Committee Member 박 경 준 (인)
Prof. Kyung-Joon Park

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#### Abstract

One of the malfunctions that can happen in the operating quadrotor is the actuator fault. The fault signal appears in the form of motor power degradation or complete loss. It can have catastrophic consequences, such as a crash, for the operating quadrotor. So, fault signal detection and fault-tolerant control are needed when a faulty situation occurs. In this regard, it is necessary to find out which actuator has failed first.

This thesis describes how to detect fault signals in actuators based on the dynamic model of quadrotors. There are two dynamic models of quadrotors used in this thesis. One is a model that assumes the quadrotor dynamics in an ideal environment. It represents the quadrotor dynamics relatively straightforward, so it is commonly used in most studies using quadrotors. The other is a model that considers the effect of aerodynamic properties generated by the rotation of propellers. Linear state equations are obtained for each two quadrotor models when the quadrotor hovers to apply the fault detection method.

The fault detection method used in this thesis is called a geometric approach, using the subspaces of the error system expressed as the estimation error of the linear system. We use the characteristics of the subspaces of the error system to design suitable filters for actuator fault detection.

To analyze the performance of the designed fault detection filter, a simulation based on MATLAB Simulink was used. We verified the performance of the designed fault detection filters and checked the effects of aerodynamic properties by the rotation of propellers on fault detection performance.


Key words :Fault Detection, Geometric approach, Unobservable Subspace, Quadrotor, Actuator, Aerodynamic Effect

부모님의 헌신과 사랑에 이 논문을 바칩니다.

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## Introduction

Drones are used in various fields, such as aerial surveillance [1], search and rescue, geological surveying, infrastructure monitoring [2], agricultural services [3] and limited delivery missions in recent years. Quadrotor is mainly used for this kind of drones, which has a simple design with four actuator rotors and highly reliable and maneuverable [4]. As research activities and application fields using quadrotor have increased, quadrotor is tested in harsh and hostile environments. Under these conditions, the quadrotor is exposed to the risk of faults. There are several types of faults that can occur in quadrotor which are known as sensor faults [5, 6], actuator faults [7-9] and propeller damages [10]. These faults interrupt the maneuver and make it difficult to accomplish their mission. In this thesis, we only handle the actuator faults, which are mostly due to motor degradation or complete loss. The occurrence of actuator fault has a critical effect on the performance of path tracking and attitude control that can lead to the crash. To avoid these faults, many studies have been contributed to fault detection and identification (FDI) methods in-flight quadrotor systems.

FDI process can be separated into two main categories. Figure 1.1 shows the whole block diagram of FDI. The first category is residue generation, which means designing fault detection filters, and the second category is to make proper decisions by using residues. In this thesis, we only deal with residue generation steps by using a geometric approach [11], which is based on linear algebraic backgrounds and the subspaces algorithms.

To use a fault detection method, we need a sophisticated model to describe the actual dynamics of the quadrotor. However, many existing studies use a classical quadrotor model. In these cases, the aerodynamic characteristics that may appear in the real envi-
ronment can cause the wrong fault alert even though there is no fault. In this regard, this thesis shows why the sophisticated quadrotor model is needed in fault detection using a geometric approach.


Figure 1.1: Block diagram of FDI.

### 1.1 Motivation

There are previous works to detect actuator faults of rotary-wing UAVs in linear [12, 13], or nonlinear cases [14,15]. Reference [15] uses RPM controller of each motor to detect actuator faults. This approach is valid when only dealing with actuator faults, but abrupt changes in motor speed can make false alarms in fault detection. To avoid this, reference 1 uses the currents of the motor as input data as well. However, these works do not include environmental disturbances such as wind gust and the aerodynamic effect of propellers. A few works $[16,17]$ introduce the sophisticated model considering the wind effect on the control of the quadrotor recently. For this reason, FDI of quadrotor actuator in the presence of aerodynamic effect and wind is currently underway. As one of the approaches for FDI of quadrotor actuator, the disturbance decoupling is to design a residue that can decouple the fault from the input signal. Various methods, such as geometric approach [18-20], unknown input observer [21] have been proposed. In particular, the geometric
approach can easily make residues when we can characterize solvability by using some algebraic properties and observability as reference [22] says.

Reference [20] uses a nonlinear geometric approach (NLGA) in quadrotor actuator fault detection with a restrictive wind effect. A detailed description of NLGA is fully introduced in [23]. In [20], only the $z$-axis direction of the wind effect is considered because the wind model is based on Blade Element Theory (BET), which divides a rotating propeller into infinitesimally small elements. In this case, the $x$-axis and $y$-axis direction of wind effect and aerodynamic properties that can affect rotating propellers are not considered. To improve the model in reference [20], we use a new wind model based on Blade Element Momentum Theory (BEMT) [17], which can describe the wind effect and aerodynamic effect more precisely than BET. We expect the detailed quadrotor model described by BEMT to distinguish whether the quadrotor is affected by a fault or aerodynamic effect.

### 1.2 Existing literature

Previous fault detection methods with a geometric approach are based on the classical quadrotor model or restricted model. Reference [19] introduced the sensor and actuator fault detection based on NLGA in classical quadrotor model. Reference [20] shows the actuator fault detection based on NLGA in a restrictive quadrotor model that can express only the $z$-axis direction of the wind. However, both references [19] and [20] are not considering the aerodynamic effect of propellers. In actual quadrotor operation, the aerodynamic effect can affect fault detection performance, and therefore a detailed model considering the aerodynamic effect is required.

### 1.3 Contribution

In this thesis, we compare the actuator fault detection performance of two different quadrotor models. One is the classical model without the aerodynamic effect and the other is the detailed model where the aerodynamic effect is taken into account. First, we design fault detection filters suitable for the classical model. Second, we show these filters do not work properly on the detailed quadrotor model due to the aerodynamic effect. Finally, we redesign the fault detection filters for the detailed model. The whole steps are
the main contribution of this thesis. Note that the effect of external wind is not taken into account in this thesis. Because when the unknown wind effect is considered, we cannot obtain the exact elements of the linear state equation of quadrotor. Also, the existing literature $[19,20]$ are working on NLGA, but it requires more skillful mathematical backgrounds and tricks, so we use the geometric approach in the linear case as a prior study first. And then, we apply the geometric approach in the nonlinear case under the limited situation.

### 1.4 Thesis outline

The outline of this thesis is as follows: Chapter 2 introduces linear algebraic backgrounds to understand the concept of a geometric approach. Definitions of $(\boldsymbol{C}, \boldsymbol{A})$-invariant subspaces and unobservable subspaces which are the core of the geometric approach are also included. Chapter 3 shows two types of quadrotor dynamics, one is a classical model and the other is a detailed model that takes into account the aerodynamic properties of propellers and external wind effect. Chapter 4 describes the algorithm to design fault detection filters. Simulation results are proposed in Chapter 5, and conclusions are shown in Chapter 6. The matrices of the linearized model in Chapter 3 are introduced in Appendix A. Additionally in Appendix B, we introduce the fault detection process in Tennessee Eastman process control system (TE-PCS) to explain the detail of the fault detection process using the geometric approach.


## Mathematical Preliminaries

In this chapter, we describe the mathematical preliminaries required to understand the fault detection process using a geometric approach. Here, the additional theorems and lemmas relating to the mathematical backgrounds are omitted because these backgrounds are not the main contents of this thesis and unnecessary proofing processes may make this chapter messy. There are various Theorems, Lemmas, and Propositions related to the geometric approach in [11,22], only the minimum mathematical concepts required for fault detection filter design problem are described here.

In Section 2.1, we define the notations used in this thesis and review some linear algebraic concepts mainly used in the geometric approach. Then $(\boldsymbol{C}, \boldsymbol{A})$-invariant subspaces and unobservable subspaces which are the core of the geometric approach to the fault detection filter design problem will be introduced in Section 2.2 and Section 2.3, respectively. The detailed process of calculating each subspace is further covered in Chapter 4.

### 2.1 Notations and backgrounds

All notations and basic mathematical backgrounds are introduced in this section. When $k$ is a positive integer, $\boldsymbol{k}$ means the set $\{1,2, \ldots, k\}$. The capital bold letters $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$, $\ldots$ denote the matrices or the vectors. Script letter $\mathcal{X}, \mathcal{Y}, \mathcal{S}, \ldots$ denote the vector spaces with the elements $x, y, s, \ldots$. The zero vector or the zero space is expressed by $\mathbf{0}$. The dimension of the vector space $\mathcal{X}$ is expressed by $d(\mathcal{X})$.

If $\mathcal{S}$ and $\mathcal{T}$ are both vector spaces, $\mathcal{S} \subseteq \mathcal{T}$ means that $\mathcal{S}$ is a subspace of $\mathcal{T}$. When $\mathcal{S}$
and $\mathcal{T}$ are subspaces of $\mathcal{X}$, then $\mathcal{S}+\mathcal{T}$ and $\mathcal{S} \cap \mathcal{T}$ are denoted as

$$
\begin{align*}
\mathcal{S}+\mathcal{T} & =\{s+t: s \in \mathcal{S}, t \in \mathcal{T}\}  \tag{2.1}\\
\mathcal{S} \cap \mathcal{T} & =\{x: x \in \mathcal{S} \text { and } x \in \mathcal{T}\}
\end{align*}
$$

$\mathcal{S}+\mathcal{T}$ means the smallest subspace contained in both $\mathcal{S}$ and $\mathcal{T}$. $\mathcal{S} \cap \mathcal{T}$ means the largest subspace containing both $\mathcal{S}$ and $\mathcal{T}$. Two subspaces $\mathcal{S}$ and $\mathcal{T}$ are called independent when $\mathcal{S} \cap \mathcal{T}=0$.

Let $\mathcal{X}$ and $\mathcal{Y}$ are called linear spaces above the field of the real number $\mathbb{R}$, then $\boldsymbol{C}: \mathcal{X} \rightarrow \mathcal{Y}$ means that map $\boldsymbol{C}$ is a linear transformation from $\mathcal{X}$ to $\mathcal{Y}$ where the vector space $\mathcal{X}$ is called the domain of $\boldsymbol{C}$, and $\mathcal{Y}$ is the codomain. The kernel of $\boldsymbol{C}$ (or nullspace of $\boldsymbol{C}$ ) is the subspace as follows

$$
\begin{equation*}
\operatorname{ker} \boldsymbol{C}=\{x: x \in \mathcal{X} \text { and } \boldsymbol{C} x=\mathbf{0}\} \subseteq \mathcal{X} \tag{2.2}
\end{equation*}
$$

The image of $\boldsymbol{C}$ is the subspace

$$
\begin{equation*}
\operatorname{Im} \boldsymbol{C}=\{y: y \in \mathcal{Y}, \exists x \in \mathcal{X} \text { and } y=\boldsymbol{C} x\} \subseteq \mathcal{Y} \tag{2.3}
\end{equation*}
$$

When $\boldsymbol{C} \in \mathbb{R}^{p \times n}, p \leq n$, if $\boldsymbol{C}$ is a full column rank, it means the rank of $\boldsymbol{C}$ is $p$ and $\operatorname{ker} \boldsymbol{C}=\mathbf{0}$.

If we consider the following linear time-invariant (LTI) system,

$$
\begin{align*}
& \dot{x}(t)=\boldsymbol{A x}(t)+\boldsymbol{B u}(t), \\
& \boldsymbol{y}(t)=\boldsymbol{C x}(t) . \tag{2.4}
\end{align*}
$$

with the maps $\boldsymbol{A}: \mathcal{X} \rightarrow \mathcal{X}, \boldsymbol{B}: \mathcal{U} \rightarrow \mathcal{X}$, and $\boldsymbol{C}: \mathcal{X} \rightarrow \mathcal{Y}(d(\mathcal{X})=n, d(\mathcal{U})=k$, $d(\mathcal{Y})=p)$, we can say that the pair of maps $(\boldsymbol{C}, \boldsymbol{A})$ is observable if

$$
\begin{equation*}
\bigcap_{i=1}^{n} \operatorname{ker}\left(\boldsymbol{C A}^{i-\mathbf{1}}\right)=\mathbf{0} . \tag{2.5}
\end{equation*}
$$

Let us derive (2.5) from the definition of observability. By definition, the system with an initial state $\boldsymbol{x}\left(\boldsymbol{t}_{\mathbf{0}}\right)$ is observable if and only if the value of the initial state can be determined from the system output $\boldsymbol{y}(\boldsymbol{t})$ that has been observed through the time interval $t_{0}<t<t_{f}$.

$$
\left[\begin{array}{c}
y\left(t_{0}\right)  \tag{2.6}\\
\dot{y}\left(t_{0}\right) \\
\ddot{y}\left(t_{0}\right) \\
\vdots \\
y^{n-1}\left(t_{0}\right)
\end{array}\right]=\left[\begin{array}{c}
C \\
C A \\
C A^{2} \\
\vdots \\
C A^{n-1}
\end{array}\right] x\left(t_{0}\right)=\mathcal{O} x\left(t_{0}\right)
$$

In order to find $\boldsymbol{x}\left(\boldsymbol{t}_{\boldsymbol{0}}\right)$, the observability matrix $\mathcal{O}$ in (2.6) should be full rank. As $\bigcap_{i=1}^{n} \operatorname{ker}\left(\boldsymbol{C A} \boldsymbol{A}^{i-\mathbf{1}}\right)$ represents $\operatorname{ker} \mathcal{O}$, when $\operatorname{ker} \mathcal{O}$ is zero, the system is observable, i.e., (2.5) is satisfied.

## $2.2(\boldsymbol{C}, \boldsymbol{A})$-invariant subspaces

The fundamental idea of geometric approach is to use the characteristics of subspaces to solve the fault detection problems. Our practical goal is to design an observer. Thus, it is essential to characterize the invariant subspaces, so the concept of a $(\boldsymbol{C}, \boldsymbol{A})$-invariant subspace is introduced.

Definition $2.1((\boldsymbol{C}, \boldsymbol{A})$-invariant subspaces). Let $\boldsymbol{A}: \mathcal{X} \rightarrow \mathcal{X}$ and $\boldsymbol{C}: \mathcal{X} \rightarrow \mathcal{Y}$. We say a subspace $\mathcal{W} \subseteq \mathcal{X}$ is $(\boldsymbol{C}, \boldsymbol{A})$-invariant if there exists an output injection map $\boldsymbol{G}: \mathcal{Y} \rightarrow \mathcal{X}$ such that

$$
\begin{equation*}
(\boldsymbol{A}+\boldsymbol{G} \boldsymbol{C}) \mathcal{W} \subseteq \mathcal{W} \tag{2.7}
\end{equation*}
$$

The family of $\boldsymbol{G}$ that satisfy (2.7) are denoted by $\mathcal{G}(\mathcal{W})$. To obtain the elements of $\mathcal{G}(\mathcal{W})$, it is necessary to find $\boldsymbol{G}$ that satisfies the following condition.

Let $\boldsymbol{W}: \mathcal{W} \rightarrow \mathcal{X}$ be the insertion map and $\boldsymbol{P}$ be a solution of maximum rank of $\boldsymbol{P W}=\mathbf{0}$. Then $\boldsymbol{G}$ is a solution of

$$
\begin{equation*}
P(A+G C) W=0 \tag{2.8}
\end{equation*}
$$

Condition (2.8) will be used as a design condition for fault detection filters in Chapter 4.

### 2.3 Unobservable subspaces

From (2.5), the unobservable subspace is introduced from following definition.
Definition 2.2 (Unobservable subspaces). Unobservable subspace of $(\boldsymbol{C}, \boldsymbol{A}), \mathcal{N} \subset \mathcal{X}$ is defined as

$$
\begin{equation*}
\mathcal{N}=\bigcap_{i=1}^{n} \operatorname{ker}\left(\boldsymbol{C} \boldsymbol{A}^{i-\mathbf{1}}\right) . \tag{2.9}
\end{equation*}
$$

If we deal with ths system $(\boldsymbol{H C}, \boldsymbol{A}+\boldsymbol{G C})$ instead of the system $(\boldsymbol{C}, \boldsymbol{A})$, where the matrices $\boldsymbol{G}: \mathcal{Y} \rightarrow \mathcal{X}$ and $\boldsymbol{H}: \mathcal{Y} \rightarrow \mathcal{Y}$ are the design factors of fault detection filter, the unobservable subspace also changes as follows

$$
\begin{equation*}
\mathcal{S}=\bigcap_{i=1}^{n} \operatorname{ker}\left(\boldsymbol{H} \boldsymbol{C}(\boldsymbol{A}+\boldsymbol{G} \boldsymbol{C})^{i-\mathbf{1}}\right) . \tag{2.10}
\end{equation*}
$$

It is obvious that unobservable subspace $\mathcal{S}$ is $(\boldsymbol{A}+\boldsymbol{G} \boldsymbol{C})$-invariant from (2.10), thus it is a $(\boldsymbol{C}, \boldsymbol{A})$-invariant subspace. $\mathcal{S}$ also satisfies the design condition (2.8), i.e., $\boldsymbol{P}(\boldsymbol{A}+\boldsymbol{G} \boldsymbol{C}) \boldsymbol{S}=$ $\mathbf{0}$, where $\mathcal{S}=\operatorname{Im} \boldsymbol{S}$. We denote the class of unobservable subspace containing subspace $\mathcal{L}$ as $\underline{\mathcal{S}}(\mathcal{L})$. From this notation, we can express the class of all unobservable subspaces of $\mathcal{X}$ as $\underline{\mathcal{S}}(\mathbf{0})$.

We can eliminate the presence of $\boldsymbol{H}$ in (2.10) by using the Propositions introduced in [11]. Let $\mathcal{S} \subset \mathcal{X}$. Then $\mathcal{S} \in \underline{\mathcal{S}}(\mathbf{0})$ if and only if there exists a map $\boldsymbol{G}: \mathcal{Y} \rightarrow \mathcal{X}$ such that

$$
\begin{equation*}
\mathcal{S}=\bigcap_{i=1}^{n}\left\{(\boldsymbol{A}+\boldsymbol{G} \boldsymbol{C})^{-(i-1)}(\operatorname{ker} \boldsymbol{C}+\boldsymbol{\mathcal { S }})\right\} . \tag{2.11}
\end{equation*}
$$

Also, if $\mathcal{S} \in \underline{\mathcal{S}}(\mathbf{0})$, then (2.11) applies to every map $\boldsymbol{G} \in \mathcal{G}(\boldsymbol{S})$. This proposition will be used later in Chapter 4.

## Quadrotor dynamics

In this thesis, we propose to detect the actuator fault of the quadrotor by using a geometric approach. To use this fault detection method, we need an observable LTI system with actuator fault inputs from the dynamic model of the quadrotor. Therefore, this thesis begins with finding the proper quadrotor model for fault detection.

In this chapter, we introduce two different quadrotor dynamics that can be used for fault detection. One is the classical quadrotor model, which is widely accepted in [4,12,13]. This classical dynamic model can relatively simply describe the quadrotor dynamics under ideal conditions, so it is used in most studies using quadrotor. The other is the quadrotor model that considers the external wind aerodynamic effects on the rotating propellers. These effects can be derived based on the Blade Element Momentum Theory (BEMT), which is normally used in helicopter dynamics. The model based on BEMT can express the precise dynamics of the propeller because it considers the structural properties of propeller blade and airflow around the propeller.

We linearize each model to apply the geometric approach for fault detection and see the fault detection performance differences between the two models.

First, we derive the widely accepted quadrotor dynamics in Section 3.1. In Section 3.2, we introduce a detailed quadrotor model which considers the aerodynamics on a rotating propeller. The force and the torque on a rotating propeller based on BEMT is described in Section 3.3. Finally, we discuss the Jacobian linearization and controller design in Section 3.4.

### 3.1 Classical quadrotor model



Figure 3.1: Coordinates and control inputs of the quadrotor.
Coordinates and control inputs of the quadrotor are shown in Figure 3.1. Basis of the inertial frame is the standard vector $\boldsymbol{e}_{\mathbf{1}}=\left[\begin{array}{lll}1 & 0 & 0\end{array}\right]^{T}, \boldsymbol{e}_{\mathbf{2}}=\left[\begin{array}{lll}0 & 1 & 0\end{array}\right]^{T}, \boldsymbol{e}_{\mathbf{3}}=\left[\begin{array}{lll}0 & 0 & 1\end{array}\right]^{T}$ and the basis of the body frame is the vector $\boldsymbol{R}_{\mathbf{1}}, \boldsymbol{R}_{\mathbf{2}}, \boldsymbol{R}_{\mathbf{3}}$. Rotational speed of each propeller is denoted by $w_{i}, i=1,2,3,4$, respectively. The force of the quadrotor is denoted by $f$ which is in the direction of $\boldsymbol{R}_{\mathbf{3}}$ and torques along the axis $\boldsymbol{R}_{\boldsymbol{i}}$ is denoted by $\tau_{j}$ for $j=1,2,3$. Unlike classical quadrotor model widely used in [4], additional force vector $f_{D} \in \mathbb{R}^{3}$ and torque vector $\boldsymbol{\tau}_{\boldsymbol{D}} \in \mathbb{R}^{3}$ which include the effect of wind and aerodynamic drag lead to the following equations:

$$
\begin{align*}
m \ddot{\boldsymbol{x}} & =-m g \boldsymbol{e}_{\boldsymbol{3}}+\boldsymbol{R}\left(f \boldsymbol{e}_{\mathbf{3}}+\boldsymbol{f}_{\boldsymbol{D}}\right), \\
\dot{\boldsymbol{R}} & =\boldsymbol{R} \hat{\boldsymbol{\Omega}},  \tag{3.1}\\
\boldsymbol{I} \dot{\boldsymbol{\Omega}} & =(\boldsymbol{I} \boldsymbol{\Omega}) \times \boldsymbol{\Omega}+\left(\boldsymbol{\tau}+\boldsymbol{\tau}_{\boldsymbol{D}}\right) .
\end{align*}
$$

Here, $\boldsymbol{x}=\left[\begin{array}{lll}x_{1} & x_{2} & x_{3}\end{array}\right]^{T}$ is the position of the quadrotor in inertial frame, $\boldsymbol{R} \in \mathbb{R}^{3 \times 3}$ is the rotation matrix that changes the coordinates of the quadrotor from body frame to inertial frame, i.e., $\boldsymbol{R}=\left[\begin{array}{lll}\boldsymbol{R}_{\mathbf{1}} & \boldsymbol{R}_{\mathbf{2}} & \boldsymbol{R}_{\mathbf{3}}\end{array}\right], \boldsymbol{\tau}=\left[\tau_{1} \tau_{2} \tau_{3}\right]^{T}$ is the torque of the quadrotor, $m$ is the mass
of the quadrotor, $g$ is the gravitational acceleration, $\boldsymbol{I} \in \mathbb{R}^{3 \times 3}$ is the moment of inertia matrix, $\boldsymbol{\Omega}=\left[\begin{array}{lll}\Omega_{1} & \Omega_{2} & \Omega_{3}\end{array}\right]^{T}$ is the angular velocity vector, and $\hat{\boldsymbol{\Omega}}$, the hat map of $\boldsymbol{\Omega}$, is the skew symmetric matrix given by

$$
\hat{\Omega}=\left[\begin{array}{ccc}
0 & -\Omega_{3} & \Omega_{2}  \tag{3.2}\\
\Omega_{3} & 0 & -\Omega_{1} \\
-\Omega_{2} & \Omega_{1} & 0
\end{array}\right]
$$

Control input $f$ and $\tau$ can be expressed by thrust of each propeller through the following equation

$$
\left[\begin{array}{c}
f  \tag{3.3}\\
\tau_{1} \\
\tau_{2} \\
\tau_{3}
\end{array}\right]=\left[\begin{array}{cccc}
1 & 1 & 1 & 1 \\
0 & l & 0 & -l \\
-l & 0 & l & 0 \\
k_{m} & -k_{m} & k_{m} & -k_{m}
\end{array}\right]\left[\begin{array}{c}
T_{1} \\
T_{2} \\
T_{3} \\
T_{4}
\end{array}\right]
$$

where $T_{i}$ is the thrust of $i$-th propeller for $i=1,2,3,4, l$ is the length from motor to center of the quadrotor, and $k_{m}$ is a positive constant. In classical quadrotor model, the effect of wind is not considered $\left(\boldsymbol{f}_{\boldsymbol{D}}=0, \boldsymbol{\tau}_{\boldsymbol{D}}=0\right)$, thrust $T_{i}$ is simply described as $T_{i}=k_{w} w_{i}{ }^{2}$ where $k_{w}$ is a positive constant.

### 3.2 Aerodynamic effect of propellers

According to BEMT, when the quadrotor is moving in the wind, the thrust $T_{i}$ is expressed in a more complex equation than $T=k_{w} w_{i}{ }^{2}$ which is used in classical model. Let the wind be represented by $\boldsymbol{W} \in \mathbb{R}^{3 \times 3}$ in inertial frame. Then, the relative velocity of the quadrotor center of mass in inertial frame can be expressed as $\boldsymbol{W}-\dot{\boldsymbol{x}}$. Define $\boldsymbol{V}_{r}$ as the relative velocity in the body frame, i.e.,

$$
\begin{equation*}
\boldsymbol{V}_{r}=\boldsymbol{R}^{T}(\boldsymbol{W}-\dot{\boldsymbol{x}}) \tag{3.4}
\end{equation*}
$$

Assume that $\boldsymbol{V}_{r}$ is the same at all four propeller. This assumption is reasonable since the spinning propeller can be regarded as a flat disk. Generally, classical quadrotor model
does not take into account the relative velocity by wind effect. This is the main difference between the classical model and detailed model. Taking into account the relative velocity $\boldsymbol{V}_{\boldsymbol{r}}$, the relative velocity experienced by the $i$-th propeller, denoted by $\boldsymbol{V}_{\boldsymbol{i}}$ is given by

$$
\begin{align*}
& V_{\mathbf{1}}=-l \hat{\Omega} e_{1}+V_{r} \\
& V_{\mathbf{2}}=-l \hat{\Omega} e_{2}+V_{r} \\
& V_{\mathbf{3}}=l \hat{\boldsymbol{\Omega}} e_{\mathbf{1}}+V_{r}  \tag{3.5}\\
& V_{\mathbf{4}}=l \hat{\boldsymbol{\Omega}} e_{\mathbf{2}}+V_{r}
\end{align*}
$$

For simple notation, let the $j$-th component of the vector $\boldsymbol{V}_{\boldsymbol{i}}$ by $V_{i, j}$, i.e.,

$$
\begin{equation*}
V_{i, j}=\boldsymbol{V}_{i}^{T} e_{j} \tag{3.6}
\end{equation*}
$$

for $i=1,2,3,4$ and $j=1,2,3$.
Assume that the first propeller moving in the air shown in Figure 3.2. Relative velocity $\boldsymbol{V}_{\boldsymbol{i}}$ can be separated into two vectors, $\boldsymbol{V}_{\boldsymbol{z}, \boldsymbol{i}}$ and $\boldsymbol{V}_{\boldsymbol{h}, \boldsymbol{i}} . \boldsymbol{V}_{\boldsymbol{z}, \boldsymbol{i}}$ is in the direction of $\boldsymbol{R}_{\boldsymbol{3}}$ and $\boldsymbol{V}_{\boldsymbol{h}, \boldsymbol{i}}$ is in the plane generated by $\boldsymbol{R}_{\mathbf{1}}$ and $\boldsymbol{R}_{\mathbf{2}}$,

$$
\begin{align*}
& \boldsymbol{V}_{z, i}=\left[\begin{array}{c}
0 \\
0 \\
V_{i, 3}
\end{array}\right], \\
& \boldsymbol{V}_{\boldsymbol{h}, \boldsymbol{i}}=\left[\begin{array}{c}
V_{i, 1} \\
V_{i, 2} \\
0
\end{array}\right] . \tag{3.7}
\end{align*}
$$

In Figure 3.2, the radius of the rotating propeller considered as a disk is denoted by $r$, and the distance between the disk and the plane defined by $\boldsymbol{R}_{\mathbf{1}}$ and $\boldsymbol{R}_{\mathbf{1}}$ passing through the center of mass of the quadrotor is denoted by $d$. Also, the induced velocity is denoted by $v_{i}$. We assume that

$$
\boldsymbol{v}_{\boldsymbol{i}}=\left[\begin{array}{c}
0  \tag{3.8}\\
0 \\
v_{i, 3}
\end{array}\right],
$$



Figure 3.2: Detailed coordinate representation of the first propeller.
with a scalar $v_{i, 3}$. This is a possible assumption because the airflow in the direction of $\boldsymbol{R}_{\mathbf{3}}$ is much more dominant and the airflow in the other direction is negligible [24].

According to BEMT, total thrust of the quadrotor is generated in two directions, where the scalar $T_{i}$ represents the thrust in the direction of $\boldsymbol{V}_{z, i}$ and the vector $\boldsymbol{H}_{\boldsymbol{i}}$ represents the thrust in the direction of $\boldsymbol{V}_{\boldsymbol{h}, \boldsymbol{i}}$. The amount of thrust $T_{i}$ generated by a rotating propeller with angular velocity $w_{i}[\mathrm{rad} / \mathrm{sec}]$ is given by

$$
\begin{equation*}
T_{i}=c_{1}\left[c_{2}\left(1+\frac{3}{2} \mu_{i}^{2}\right)-\lambda_{i}\right] w_{i}^{2}, \tag{3.9}
\end{equation*}
$$

where $c_{1}$ and $c_{2}$ are positive constants. The advanced ratio $\mu_{i}$ and the inflow ratio $\lambda_{i}$ are given by

$$
\begin{align*}
\mu_{i} & =\frac{\left\|\boldsymbol{V}_{\boldsymbol{h}, i}\right\|}{r w_{i}}, \\
\lambda_{i} & =\frac{-V_{i, 3}+v_{i, 3}}{r w_{i}} . \tag{3.10}
\end{align*}
$$

Now, considering the component of induced velocity $v_{i, 3}$ is a complicated task. In this thesis, we use the approach of [24], where $v_{i, 3}$ is approximated by a linear combination of $w_{i}$ and $V_{i, 3}$. This approximation is based on experiment data in [17]. Then $v_{i}$ can be written as

$$
\begin{equation*}
v_{i, 3}=a w_{i}+b V_{i, 3}, \tag{3.11}
\end{equation*}
$$

where $a$ and $b$ are positive constants.
Using (3.9), (3.10), and (3.11), we can denote $T_{i}$ as follows

$$
\begin{equation*}
T_{i}=k_{w} w_{i}^{2}+k_{z} \boldsymbol{V}_{z, i}^{\boldsymbol{T}} \boldsymbol{e}_{\mathbf{3}} w_{i}+k_{h}\left\|\boldsymbol{V}_{\boldsymbol{h}, \boldsymbol{i}}\right\|^{2} . \tag{3.12}
\end{equation*}
$$

where $k_{w}, k_{z}$, and $k_{h}$ are positive constants. The total thrust $T$ is the sum of the thrust of each propeller $T_{i}$, i.e., $T=\sum_{i=1}^{4} T_{i}$.

The horizontal thrust $H_{i}$ means the effect of aerodynamic drag such as blade flapping, induced drag, etc. If the state of quadrotor changes slowly, $H_{i}$ can be denoted by

$$
\begin{equation*}
\boldsymbol{H}_{\boldsymbol{i}}=c T_{i} \boldsymbol{V}_{\boldsymbol{h}, \boldsymbol{i}}, \tag{3.13}
\end{equation*}
$$

where the constant $c$ is positive. The estimation of all constant in (3.3), (3.12), and (3.13) is deeply discussed in [17].

### 3.3 Modeling of the drag force and drag torque

In (3.1), drag force $f_{D}$ is expressed by $H_{i}, i=1,2,3,4$ in (3.13) as $f_{D}=\sum_{i=1}^{4} H_{i}$. By using (3.12) and (3.13), we can express $f_{D}$ as

$$
\boldsymbol{f}_{D}=c T\left[\begin{array}{c}
\boldsymbol{V}_{r}^{T} \boldsymbol{e}_{1}  \tag{3.14}\\
\boldsymbol{V}_{r}^{T} \boldsymbol{e}_{2} \\
0
\end{array}\right]+c l \Omega_{3}\left[\begin{array}{c}
T_{2}-T_{4} \\
T_{3}-T_{1} \\
0
\end{array}\right]
$$

Also, drag torque $\tau_{D, i}$ produced by each propeller is as follows

$$
\begin{align*}
& \boldsymbol{\tau}_{D, \mathbf{1}}=\left[\begin{array}{l}
l \\
0 \\
d
\end{array}\right] \times \boldsymbol{H}_{\mathbf{1}}, \\
& \boldsymbol{\tau}_{D, \mathbf{2}}=\left[\begin{array}{l}
0 \\
l \\
d
\end{array}\right] \times \boldsymbol{H}_{\mathbf{2}},  \tag{3.15}\\
& \boldsymbol{\tau}_{D, \mathbf{3}}=\left[\begin{array}{c}
-l \\
0 \\
d
\end{array}\right] \times \boldsymbol{H}_{\mathbf{3}}, \\
& \boldsymbol{\tau}_{D, 4}=\left[\begin{array}{c}
0 \\
-l \\
d
\end{array}\right] \times \boldsymbol{H}_{\mathbf{4}},
\end{align*}
$$

Then, the total drag torque $\tau_{D}$ in (3.1) can be written as

$$
\begin{align*}
\tau_{D} & =\tau_{D, 1}+\tau_{D, 2}+\tau_{D, 3}+\tau_{D, 4} \\
& =d c T\left[\begin{array}{c}
-\boldsymbol{V}_{\boldsymbol{r}}{ }^{T} \boldsymbol{e}_{\mathbf{2}} \\
\boldsymbol{V}_{\boldsymbol{r}}^{\boldsymbol{T}} \boldsymbol{e}_{\mathbf{1}} \\
0
\end{array}\right]+\operatorname{dcl\Omega _{3}}\left[\begin{array}{c}
T_{1}-T_{3} \\
T_{2}-T_{4} \\
0
\end{array}\right]+c l\left[\begin{array}{c}
0 \\
0 \\
T_{1} V_{1,2}-T_{2} V_{2,1}-T_{3} V_{3,2}+T_{4} V_{4,1}
\end{array}\right] \tag{3.16}
\end{align*}
$$

Completely, we explained the quadrotor dynamics in a wind field by (3.1), (3.3), (3.4), (3.5), (3.7), (3.12), (3.14), and (3.16). Note that the wind $\boldsymbol{W}$ is not the only difference between the classical quadrotor model and the detailed quadrotor model using BEMT. In detailed model, although the wind $\boldsymbol{W}$ is zero, the thrust $T_{i}$ in (3.12) does not change as $T_{i}=k_{w} w_{i}{ }^{2}$. Also, $f_{D}$ and $\tau_{D}$ is not equal to zero as well. This is because the detailed model considers the aerodynamics of the propeller and the air which depends on the dynamics and attitude of the quadrotor as well as the wind $\boldsymbol{W}$. This can be proved by $V_{r}, V_{1}, V_{2}$, $V_{3}$, and $V_{4}$ in (3.4), (3.5). These terms depend on $\boldsymbol{R}, \dot{\boldsymbol{x}}$, and $\boldsymbol{\Omega}$ as well as the wind $\boldsymbol{W}$.

### 3.4 Linearization and controller design

We need a linearized model to apply the fault detection method which is called geometric approach to the quadrotor.For this, combine the quadrotor dynamics in (3.1), (3.12), (3.14), and (3.16) as

$$
\begin{equation*}
\dot{\boldsymbol{x}}=F(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{W}) . \tag{3.17}
\end{equation*}
$$

Here, $\boldsymbol{x} \in \mathbb{R}^{12}$ is state vector and $\boldsymbol{u} \in \mathbb{R}^{4}$ is input vector as follows

$$
\begin{align*}
\boldsymbol{x} & =\left[\begin{array}{lll}
x_{3} \dot{x}_{3} x_{1} \dot{x_{1}} \theta \Omega_{2} x_{2} \dot{x_{2}} \phi \Omega_{1} \psi \Omega_{3}
\end{array}\right]^{T},  \tag{3.18}\\
\boldsymbol{u} & =\left[\begin{array}{lll}
w_{1}^{2} & w_{2}^{2} & w_{3}^{2} \\
{ }^{2} & w_{4}^{2}
\end{array}\right]^{T},
\end{align*}
$$

where $\phi, \theta, \psi$ are roll, pitch, yaw angles, respectively. When we consider the hovering situation, the operating point $\boldsymbol{x}^{*}$ and $\boldsymbol{u}^{*}$ can be expressed by

$$
\left.\begin{array}{l}
\boldsymbol{x}^{*}=\left[\begin{array}{llllllllll}
x_{3}{ }^{*} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} 0\right.
\end{array}\right]^{T},
$$

Then, the state matrix $\boldsymbol{A}$ and input matrix $\boldsymbol{B}$ of the linear model are denoted by

$$
\begin{align*}
\boldsymbol{A} & =\left.\frac{\partial F(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{W})}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\boldsymbol{x}^{*}, \boldsymbol{u}=\boldsymbol{u}^{*}} \in \mathbb{R}^{12 \times 12}, \\
\boldsymbol{B} & =\left.\frac{\partial F(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{W})}{\partial \boldsymbol{u}}\right|_{\boldsymbol{x}=\boldsymbol{x}^{*}, \boldsymbol{u}=\boldsymbol{u}^{*}} \in \mathbb{R}^{12 \times 4}, \tag{3.20}
\end{align*}
$$

and the linear state equation can be written as

$$
\begin{align*}
\dot{\boldsymbol{x}} & =\boldsymbol{A} \boldsymbol{x}+\boldsymbol{B} \boldsymbol{u} \\
& =\left(\boldsymbol{A}_{\mathbf{0}}+\overline{\boldsymbol{A}}\right) \boldsymbol{x}+\left(\boldsymbol{B}_{\mathbf{0}}+\overline{\boldsymbol{B}}\right) \boldsymbol{u}, \tag{3.21}
\end{align*}
$$

where the matrix $\boldsymbol{A}_{\mathbf{0}}$ and $\boldsymbol{B}_{\mathbf{0}}$ are irrelevant to $\boldsymbol{W}$, compared to the elements of the matrix $\overline{\boldsymbol{A}}$ and $\overline{\boldsymbol{B}}$ are related to $\boldsymbol{W}$. When $\boldsymbol{W}$ is zero, $\overline{\boldsymbol{A}}$ and $\overline{\boldsymbol{B}}$ equal to zero. All matrices in (3.21) are detailed in Appendix A.

In this thesis, we assume that there is no effect derived by the external wind $\boldsymbol{W}$, i.e., $\overline{\boldsymbol{A}}=\mathbf{0}$ and $\overline{\boldsymbol{B}}=\mathbf{0}$. We need a precise linear model to adopt a geometric approach, which means we should have all elements of the matrices in (3.21). This means that we have to know the exact size of $\boldsymbol{W}$ to detect fault through a geometric approach in wind conditions. However, it is hard work and if the size of $\boldsymbol{W}$ changes in real time, it is almost impossible to redesign the linear model accordingly.

Now, around the hovering position, we use a LQR controller for controllable matrix pair $\left(\boldsymbol{A}_{\mathbf{0}}, \boldsymbol{B}_{\mathbf{0}}\right)$ such that

$$
\begin{equation*}
\boldsymbol{u}=-\boldsymbol{K}\left(\boldsymbol{x}-\boldsymbol{x}^{*}\right) . \tag{3.22}
\end{equation*}
$$

Since the designing controller is not the subject covered in this thesis, see [17] for more information. LQR parameters $\boldsymbol{P}$ for $\boldsymbol{x}$ and $\boldsymbol{Q}$ for $\boldsymbol{u}$ is given as

$$
\begin{align*}
& \boldsymbol{P}=\operatorname{diag}\left(10^{6}, 10^{3}, 10^{6}, 10^{3}, 10^{6}, 10^{3}, 10^{6}, 10^{3}, 10^{6}, 10^{3}, 10^{6}, 10^{3}\right) \\
& \boldsymbol{Q}=\operatorname{diag}\left(10^{-8}, 10^{-8}, 10^{-8}, 10^{-8}\right) \tag{3.23}
\end{align*}
$$

## 4

## Fault Detection Problems

In Chapter 3, we discussed two different types of the quadrotor model. One is the classical model and the other is the detailed model where the wind effect and aerodynamic drag are considered. In addition, the process of obtaining a linear model through Jacobian linearization to apply fault detection through geometric approach was also described. We have confirmed that the linear state equation of the classical quadrotor model and the detailed quadrotor model are different. Now, we can now follow the fault detection process with two given linear models.

The main contributions of this thesis are included in this chapter. We shall formulate and solve the fault detection problem with a linear state equation of the quadrotor obtained in Chapter 3. We use a geometric approach as the fault detection method which is mainly covered in [11].

In Section 4.1, we introduce how fault detection using geometric approach is done in the linear model. Specific procedures are listed in subsections. First, we start with how to obtain $(C, A)$-invariant subspaces and unobservable subspaces from each linear state equation. Through the recursive algorithm in [11], we can get each subspace. The process of obtaining the observer gain $\boldsymbol{G}_{\boldsymbol{i}}$ and the filter matrix $\boldsymbol{H}_{\boldsymbol{i}}$ will be followed. The special case that can simplify the process of designing fault detection filter is covered in Section 4.2. Section 4.3 and 4.4 will show in detail the process of generating four residues with linear state equation of classic quadrotor model and detailed quadrotor model, respectively.

### 4.1 Fault detection problems

Let us consider an observable LTI system with $k$ actuator inputs:

$$
\begin{align*}
& \dot{\boldsymbol{x}}(\boldsymbol{t})=\boldsymbol{A} \boldsymbol{x}(\boldsymbol{t})+\boldsymbol{B} \boldsymbol{u}(\boldsymbol{t})+\sum_{i=1}^{k} \boldsymbol{L}_{i} m_{i}(t),  \tag{4.1}\\
& \boldsymbol{y}(\boldsymbol{t})=\boldsymbol{C} \boldsymbol{x}(\boldsymbol{t})
\end{align*}
$$

Here, $\boldsymbol{x}(\boldsymbol{t}) \in \mathbb{R}^{n}$ is the state vector, $\boldsymbol{u}(\boldsymbol{t}) \in \mathbb{R}^{k}$ is the input vector, and $\boldsymbol{y}(\boldsymbol{t}) \in \mathbb{R}^{p}$ is the output vector. Dimension of the matrix in state space form is $\boldsymbol{A} \in \mathbb{R}^{n \times n}, \boldsymbol{B} \in \mathbb{R}^{n \times k}$, and $\boldsymbol{C} \in \mathbb{R}^{p \times n} . \boldsymbol{L}_{\boldsymbol{i}}$ is the $i$-th column of the input matrix $\boldsymbol{B}$, and the term $\boldsymbol{L}_{\boldsymbol{i}} m_{i}(t)$ represents a fault of the $i$-th actuator. We assume the functions $m_{i}(t)$ are completely unknown. However, when there is no error in $i$-th actuator, $m_{i}(t)=0$ by definition. Also, we only deal with actuator faults in this thesis, we assume our sensors are perfectly reliable.

Consider that we design a full-order observer with the following structure for the system in (4.1).

$$
\begin{align*}
& \dot{\hat{x}}(t)=A \hat{x}(t)+B u(t)-G_{i}(y(t)-C \hat{x}(t)), \\
& \hat{y}(t)=C \hat{x}(t) \tag{4.2}
\end{align*}
$$

In (4.2), $\hat{\boldsymbol{x}}(\boldsymbol{t}) \in \mathbb{R}^{n}$ is the estimated state vector, $\hat{\boldsymbol{y}}(\boldsymbol{t}) \in \mathbb{R}^{p}$ is a pseudo measurement. If there is no fault signal and the observer is stable, there would be no prediction error between $\boldsymbol{y}(\boldsymbol{t})$ and $\hat{\boldsymbol{y}}(\boldsymbol{t})$. However, if the $i$-th actuator fails, i.e., $m_{i}(t) \neq 0$, the observer continues to predict the nominal output of the system while the actual system makes the faulty output. Afterward, we use the directional properties of the prediction error to identify the actuator fault. Let us define $k$ different linear transformations, $r_{1}(t), r_{2}(t)$, $\ldots, r_{k}(t)$, as follows

$$
\begin{align*}
r_{1}(t) & =\boldsymbol{H}_{\mathbf{1}}(\boldsymbol{y}(\boldsymbol{t})-\hat{\boldsymbol{y}}(\boldsymbol{t})), \\
r_{2}(t) & =\boldsymbol{H}_{\mathbf{2}}(\boldsymbol{y}(\boldsymbol{t})-\hat{\boldsymbol{y}}(\boldsymbol{t})),  \tag{4.3}\\
\vdots & \\
r_{k}(t) & =\boldsymbol{H}_{k}(\boldsymbol{y}(\boldsymbol{t})-\hat{\boldsymbol{y}}(\boldsymbol{t})) .
\end{align*}
$$

If we can design the proper matrices $\boldsymbol{G}_{\boldsymbol{i}}, \boldsymbol{H}_{\mathbf{1}}, \boldsymbol{H}_{\mathbf{2}}, \ldots$, and $\boldsymbol{H}_{\boldsymbol{k}}$ such that the fault of the $i$-th actuator only shows up in $r_{i}(t)$, fault detection problem will be solved easily. When we define $\boldsymbol{e}(\boldsymbol{t})=\boldsymbol{x}(\boldsymbol{t})-\hat{\boldsymbol{x}}(\boldsymbol{t})$, these matrices can be designed by using subspace properties of the following system,

$$
\begin{align*}
\dot{\boldsymbol{e}}(\boldsymbol{t}) & =\left(\boldsymbol{A}+\boldsymbol{G}_{i} \boldsymbol{C}\right) \boldsymbol{e}(\boldsymbol{t})+\sum_{i=1}^{k} \boldsymbol{L}_{i} m_{i}(t), \\
r_{1}(t) & =\boldsymbol{H}_{\mathbf{1}} \boldsymbol{C e}(\boldsymbol{t}) \\
r_{2}(t) & =\boldsymbol{H}_{\mathbf{2}} \boldsymbol{C e}(\boldsymbol{t})  \tag{4.4}\\
\vdots & \\
r_{k}(t) & =\boldsymbol{H}_{\boldsymbol{k}} \boldsymbol{C e}(\boldsymbol{t})
\end{align*}
$$

In (4.4), suppose we have two actuators, i.e., $k=2$. If we want to make first residue $r_{1}(t)$ detect only the fault of first actuator, for a nonzero $m_{2}(t)$ not to affect $r_{1}(t)$, the image of $\boldsymbol{L}_{\mathbf{2}}$ should be in the unobservable subspace of the system $\left(\boldsymbol{H}_{\mathbf{1}} \boldsymbol{C}, \boldsymbol{A}+\boldsymbol{G}_{\mathbf{1}} \boldsymbol{C}\right)$. Also for a nonzero $m_{1}(t)$ to show up in $r_{1}(t)$, the image of $\boldsymbol{L}_{\mathbf{1}}$ should not intersect the unobservable subspace of the system $\left(\boldsymbol{H}_{1} \boldsymbol{C}, \boldsymbol{A}+\boldsymbol{G}_{1} \boldsymbol{C}\right)$. This concept applies equally to second residue $r_{2}(t)$.

In this case, we can customize the observability properties the system (4.4) by designing proper matrices $\boldsymbol{G}_{\mathbf{1}}, \boldsymbol{G}_{\mathbf{2}}, \boldsymbol{H}_{\mathbf{1}}$, and $\boldsymbol{H}_{\mathbf{2}}$. Obviously, the unobservable subspace of $\left(\boldsymbol{H}_{\mathbf{1}} \boldsymbol{C}, \boldsymbol{A}+\right.$ $\left.\boldsymbol{G}_{\mathbf{1}} \boldsymbol{C}\right)$ is the subspace spanned by eigenvectors of $\boldsymbol{A}+\boldsymbol{G}_{\mathbf{1}} \boldsymbol{C}$ which are in the null space of $\boldsymbol{H}_{\mathbf{1}} \boldsymbol{C}$. Also, the column vector $\boldsymbol{L}_{\mathbf{2}}$ should be expressed as a linear combination of these eigenvectors because the second actuator fault should not show up in the first residue. Thus, fault detection problem in this thesis is to use the freedom in assigning the eigenvectors of $\boldsymbol{A}+\boldsymbol{G}_{\mathbf{1}} \boldsymbol{C}$ that satisfies the fault detection conditions.

On the other hand, in geometric approach, our goal is to find the existence of subspaces $\mathcal{S}_{1}$ and $\mathcal{S}_{2}$ which contain the images of $\boldsymbol{L}_{\mathbf{2}}$ and $\boldsymbol{L}_{\mathbf{1}}$, respectively. This approach is easier than obtaining the matrices $\boldsymbol{G}_{\mathbf{1}}, \boldsymbol{G}_{\mathbf{2}}, \boldsymbol{H}_{\mathbf{1}}$, and $\boldsymbol{H}_{\mathbf{2}}$ directly because if such subspaces $\mathcal{S}_{1}$ and $\mathcal{S}_{2}$ exist and can be obtained from the matrices $\boldsymbol{A}, \boldsymbol{B}$, and $\boldsymbol{C}$, then we can easily find the matrices $\boldsymbol{G}_{\mathbf{1}}, \boldsymbol{G}_{\mathbf{2}}, \boldsymbol{H}_{\mathbf{1}}$, and $\boldsymbol{H}_{\mathbf{2}}$ from $\mathcal{S}_{1}$ and $\mathcal{S}_{2}$. This is the main concept of the fault detection by using geometric approach. The following subsections introduce how to obtain the subspaces and matrices that are required for fault detection filter design by
step by step.

### 4.1.1 Algorithm for $(\boldsymbol{C}, \boldsymbol{A})$-invariant subspaces

The recursive algorithm for obtaining $(\boldsymbol{C}, \boldsymbol{A})$-invariant subspaces is presented in [11]. This algorithm is called CAISA which is literally short for $(\boldsymbol{C}, \boldsymbol{A})$-invariant subspace algorithm. Note that this thesis does not explain how the algorithm is induced. This algorithm is induced using the characteristics of $(\boldsymbol{C}, \boldsymbol{A})$-invariant subspaces and the detailed induction process is described in [11]. Through CAISA, we can find an infimal element of $(\boldsymbol{C}, \boldsymbol{A})$ invariant subspaces that includes $\mathcal{L}$, which is the image of $\boldsymbol{L}$. We denote the family of $(\boldsymbol{C}, \boldsymbol{A})$-invariant subspaces contained in $\mathcal{L}$ as $\underline{\mathcal{W}}(\mathcal{L})$.

Theorem 4.1. [11] $[(\boldsymbol{C}, \boldsymbol{A})$-invariant subspace algorithm $]$ Let $\mathcal{L} \in \mathcal{X}$ and $\mathcal{W}^{*}=\inf$ $\underline{\mathcal{W}}(\mathcal{L})$. Then $\mathcal{W}^{*}=\lim _{k \rightarrow \infty} \mathcal{W}^{k}$ where $\mathcal{W}^{k}$ satisfies the following recursion

$$
\begin{align*}
\mathcal{W}^{k+1} & =\mathcal{L}+\boldsymbol{A}\left(\mathcal{W}^{k} \cap \operatorname{ker} \boldsymbol{C}\right),  \tag{4.5}\\
\mathcal{W}^{0} & =0 .
\end{align*}
$$

CAISA can be described in terms of matrices. Let $\operatorname{Im} \boldsymbol{L}=\mathcal{L}$ and $\boldsymbol{P}_{W}^{k}$ is a projection matrix that satisfies $\boldsymbol{P}_{\boldsymbol{W}}^{\boldsymbol{k}} \boldsymbol{W}^{\boldsymbol{k}}=\mathbf{0}$ with maximum rank. When the initial condition is $\boldsymbol{W}^{\mathbf{0}}=\mathbf{0}$, we can solve the following equation recursively.

$$
\begin{align*}
{\left[\begin{array}{c}
P_{W}^{k} \\
C
\end{array}\right] T_{W}^{k} } & =0,  \tag{4.6}\\
W^{k+1} & =\left[\begin{array}{ll}
L & A T_{W}^{k}
\end{array}\right] .
\end{align*}
$$

When Rank $\boldsymbol{W}^{\boldsymbol{k + 1}}=\operatorname{Rank} \boldsymbol{W}^{\boldsymbol{k}}$, CAISA is stopped. Then, $(\boldsymbol{C}, \boldsymbol{A})$-invariant subspaces is the image of $\boldsymbol{W}^{k}$, i.e., $\mathcal{W}^{*}=\operatorname{Im} \boldsymbol{W}^{k}$. Clearly, the algorithm should converge for $k \leq n$.

### 4.1.2 Algorithm for unobservable subspaces

( $\boldsymbol{C}, \boldsymbol{A}$ )-invariant subspaces, we obtained in 4.1.1, is used to calculate unobservable subspaces. This algorithm is called UOSA for short. As in CAISA, this thesis does not explain how UOSA is induced. Through UOSA, we can find an infimal element of unobservable
subspaces that includes $\mathcal{L}$. We denote the family of unobservable subspaces contained in $\mathcal{L}$ as $\underline{\mathcal{S}}(\mathcal{L})$.

Theorem 4.2. [11] [Unobservable subspace algorithm] Let $\mathcal{L} \in \mathcal{X}, \mathcal{W}^{*}=\inf \underline{\mathcal{W}}(\mathcal{L})$, and $\mathcal{S}^{*}=\inf \underline{\mathcal{S}}(\mathcal{L})$. Then $\mathcal{S}^{*}=\lim _{k \rightarrow \infty} \mathcal{S}^{k}$ where $\mathcal{S}^{k}$ satisfies the following recursion

$$
\begin{align*}
\mathcal{S}^{k+1} & =\mathcal{W}^{*}+\left(\boldsymbol{A}^{-1} \mathcal{S}^{k}\right) \cap \operatorname{ker} \boldsymbol{C},  \tag{4.7}\\
\mathcal{S}^{0} & =\mathcal{X}
\end{align*}
$$

UOSA can also be expressed in terms of matrices. Let $\operatorname{Im} \boldsymbol{W}^{*}=\mathcal{W}^{*}$ and $\boldsymbol{P}_{\boldsymbol{S}}^{\boldsymbol{k}}$ is a projection matrix that satisfies $\boldsymbol{P}_{S}^{k} \boldsymbol{S}^{\boldsymbol{k}}=\mathbf{0}$ with maximum rank. When the initial condition is $\boldsymbol{S}^{\mathbf{0}}=\boldsymbol{I}_{\boldsymbol{n} \times \boldsymbol{n}}$, we can solve the following equation recursively.

$$
\begin{align*}
{\left[\begin{array}{c}
P_{S}^{k} A \\
C
\end{array}\right] T_{S}^{k} } & =0  \tag{4.8}\\
S^{k+1} & =\left[\begin{array}{ll}
W^{*} & T_{S}^{k}
\end{array}\right]
\end{align*}
$$

When Rank $\boldsymbol{S}^{k+1}=\operatorname{Rank} \boldsymbol{S}^{\boldsymbol{k}}$, UOSA is stopped. Then, unobservable subspaces is the image of $\boldsymbol{S}^{k}$, i.e., $\mathcal{S}^{*}=\operatorname{Im} \boldsymbol{S}^{k}$. Also, the algorithm should converge for $k \leq n$.

### 4.1.3 Solvability condition

As mentioned in Section 4.1, we can obtain unobservable subspace through CAISA and UOSA. However, before designing $G$ and $H$, we need to check the solvability condition of unobservable subspace that we obtained. Let us denote the infimal unobservable subspace that includes all $\mathcal{L}_{j},(j=1,2, \ldots, k)$ except $j \neq i,(i \leq k)$ as $\mathcal{S}_{i}^{*}$. Also, the infimal $(\boldsymbol{C}, \boldsymbol{A})$ invariant subspace that is included in $\mathcal{S}_{i}^{*}$ is denoted as $\mathcal{W}_{i}^{*}$.

Suppose we deal with two actuators, i.e., $k=2$ in (4.4). This means we can obtain two unobservable subspaces from the algorithms above. One is the infimal unobservable subspace that includes $\mathcal{L}_{2}$, i.e., $\mathcal{S}_{1}^{*}$, and the other is the infimal unobservable subspace that includes $\mathcal{L}_{1}$, i.e., $\mathcal{S}_{2}^{*}$. To solve the fault detection problem, $\mathcal{S}_{1}^{*}$ should not have a intersection with $\mathcal{L}_{1}$. Also, there should be no intersection between $\mathcal{S}_{2}^{*}$ and $\mathcal{L}_{2}$.

Theorem 4.3. [11] [Solvability condition] To solve the fault detection problem, the infimal unobservable subspace $\mathcal{S}_{i}^{*}=\inf \underline{\mathcal{S}}\left(\mathcal{L}_{i}\right)$ should satisfy the following condition

$$
\begin{equation*}
\mathcal{S}_{i}^{*} \cap \mathcal{L}_{i}=0 . \tag{4.9}
\end{equation*}
$$

All elements of unobservable subspace $\mathcal{S}_{i}^{*}$ will not appear in residue $r_{i}(t)$, if the solvability condition is not satisfied, we cannot design residue $r_{i}(t)$ that can detect the $i$-th actuator fault signal.

### 4.1.4 Design condition of the observer gain $G_{i}$

If the unobservable subspace $\mathcal{S}_{i}^{*}$ satisfies the solvability condition, now we move on to designing the observer gain $\boldsymbol{G}_{\boldsymbol{i}}$. Based on (2.8), map $\boldsymbol{G}_{\boldsymbol{i}}$ has to satisfy the following design condition,

$$
\begin{equation*}
\boldsymbol{P}_{i}\left(\boldsymbol{A}+\boldsymbol{G}_{i} \boldsymbol{C}\right) \boldsymbol{S}_{i}=0 \tag{4.10}
\end{equation*}
$$

where $\mathcal{S}_{i}^{*}=\operatorname{Im} \boldsymbol{S}_{\boldsymbol{i}}$ and $\boldsymbol{P}_{\boldsymbol{i}}$ is a projection matrix that satisfies $\boldsymbol{P}_{\boldsymbol{i}} \boldsymbol{S}_{\boldsymbol{i}}=0$ with maximum rank. In (4.10), matrices $\boldsymbol{A}$ and $\boldsymbol{C}$ are given in (4.1) and we already know $\boldsymbol{P}_{\boldsymbol{i}}$ and $\boldsymbol{S}_{\boldsymbol{i}}$ from UOSA. Thus, we can obtain $\boldsymbol{G}_{\boldsymbol{i}}$ that satisfies (4.10). Note that not all elements of $\boldsymbol{G}_{\boldsymbol{i}}$ participate in the above design conditions. Some of the elements of $\boldsymbol{G}_{\boldsymbol{i}}$ can have degrees of freedom, which is used to satisfy another design condition. In (4.4), when the residue detects the fault signal, $\boldsymbol{A}+\boldsymbol{G}_{\boldsymbol{i}} \boldsymbol{C}$ must be Hurwitz or at least marginally stable. If not, the residue will fly away. Therefore, the process of designing $\boldsymbol{G}_{\boldsymbol{i}}$ is to first define the elements of $\boldsymbol{G}_{\boldsymbol{i}}$ that satisfy the design condition (4.10), and then use the degrees of freedom to satisfy the stability condition.

### 4.1.5 Design condition of the filter matrix $\boldsymbol{H}_{i}$

In Chapter 2, the filter matrix $\boldsymbol{H}_{i}$ is needed to obtain the unobservable subspaces of system $\left(\boldsymbol{H}_{\boldsymbol{i}} \boldsymbol{C}, \boldsymbol{A}+\boldsymbol{G}_{\boldsymbol{i}} \boldsymbol{C}\right)$. However, from (2.10) and (2.11) in Chapter 2, we can get the relationship between $\mathcal{S}_{i}^{*}$ and $\boldsymbol{H}_{i}$ as follows

$$
\begin{equation*}
\operatorname{ker} \boldsymbol{H}_{i} \boldsymbol{C}=\operatorname{ker} \boldsymbol{C}+\mathcal{S}_{i}^{*} \tag{4.11}
\end{equation*}
$$

The condition (4.11) means that the filter matrix $\boldsymbol{H}_{\boldsymbol{i}}$ can be obtained from the kernel of the matrix $\boldsymbol{C}$ and unobservable subspace $\mathcal{S}_{i}^{*}$. Since the matrix $\boldsymbol{H}_{i}$ is used in residue $r_{i}(t)=\boldsymbol{H}_{\boldsymbol{i}} \boldsymbol{C e}(\boldsymbol{t}), \boldsymbol{H}_{\boldsymbol{i}}$ is expressed as one row vector to obtain one residue signal.

### 4.2 Speicial case when $C$ is full rank

In general case, we have to find the infimal $(\boldsymbol{C}, \boldsymbol{A})$-invariant subspace $\mathcal{W}_{i}^{*}$ and infimal unobservable subspace $\mathcal{S}_{i}^{*}$ through CAISA and UOSA introduced in Section 4.1. However, when $\boldsymbol{C}$ is full rank, we can easily solve the infimality problem. In CAISA, $\operatorname{ker} \boldsymbol{C}=\mathbf{0}$ and $\mathcal{W}_{i}^{*}$ is simply obtained as a set of $\mathcal{L}_{j},(j=1,2, \ldots, k)$ except $j \neq i,(i \leq k)$. Thus, $\mathcal{S}_{i}^{*}$ is also same as $\mathcal{W}_{i}^{*}$ in UOSA. In fact, it is uncommon that $\boldsymbol{C}$ is full rank in actual case. However, for the convenience of calculation, we used $\boldsymbol{C}$ as an identity matrix in the linear model of quadrotor. For the case where $\boldsymbol{C}$ is not full rank, we describe the fault detection process in a system called Tennessee-Eastman process control system (TE-PCS) in Appendix B.

### 4.3 Designing fault detection filters in classical quadrotor model

In this section, we describe how to design the observer gain $\boldsymbol{G}_{\boldsymbol{i}}$ and the filter matrix $\boldsymbol{H}_{\boldsymbol{i}}$ in the linear state equation of classical quadrotor model. Since the quadrotor has four motors, four actuator fault detection filters are needed.

First, we design $\boldsymbol{G}_{\mathbf{1}}$ and $\boldsymbol{H}_{\mathbf{1}}$ to build the residue $r_{1}(t)$ that can detect only the fault of the first actuator. As we assume that $\boldsymbol{C}$ is full rank, $\boldsymbol{S}_{\mathbf{1}}$ can be obtained as $\left[\begin{array}{lll}\boldsymbol{L}_{\mathbf{2}} & \boldsymbol{L}_{\mathbf{3}} & \boldsymbol{L}_{4}\end{array}\right]$. We already know the specific values of each element in $\boldsymbol{S}_{\mathbf{1}}$, which are expressed in (4.12). Note that all elements of $\boldsymbol{S}_{\mathbf{1}}$ are expressed in rounded form to the fourth decimal place. From $\boldsymbol{S}_{\mathbf{1}}$, we can derive the projection matrix $\boldsymbol{P}_{\mathbf{1}}$ that satisfies $\boldsymbol{P}_{\mathbf{1}} \boldsymbol{S}_{\mathbf{1}}=\mathbf{0}$. Since the three columns of $\boldsymbol{S}_{\mathbf{1}}$ are linearly independent, $\boldsymbol{P}_{\mathbf{1}}$ has nine rows to be a maximum rank in (4.13).

$$
\begin{align*}
& \boldsymbol{S}_{\mathbf{1}}=10^{-5} \times\left[\begin{array}{ccc}
0 & 0 & 0 \\
0.0145 & 0.0145 & 0.0145 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0.3277 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0.3259 & 0 & -0.3259 \\
0 & 0 & 0 \\
-0.6987 & 0.6987 & -0.6987
\end{array}\right],  \tag{4.12}\\
& \boldsymbol{P}_{\mathbf{1}}=\left[\begin{array}{cccccccccccc}
0 & 0.6987 & 0 & 0 & 0 & -0.0619 & 0 & 0 & 0 & 0 & 0 & 0.0145 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right] . \tag{4.13}
\end{align*}
$$

Now, we can design $\boldsymbol{G}_{\boldsymbol{1}}$ that satisfies (4.10). Let the elements of $\boldsymbol{G}_{\mathbf{1}}$ in $i$-th row and $j$-th column as $g_{1}(i, j),(i \in \mathbf{1 2}, j \in \mathbf{1 2})$. The elements of $\boldsymbol{G}_{\mathbf{1}}$ that are needed to satisfy the design condition are selected in (4.14). The elements whose values are not determined yet mean the degrees of freedom of $\boldsymbol{G}_{\mathbf{1}}$. These degrees of freedom are used to make that $\left(\boldsymbol{A}+\boldsymbol{G}_{\mathbf{1}} \boldsymbol{C}\right)$ is Hurwitz. Finally, $\boldsymbol{G}_{\mathbf{1}}$ with all elements are determined is expressed in (4.15).

$$
\begin{array}{llll}
g_{1}(1,2)=-1, & g_{1}(1,6)=0, & & g_{1}(1,10)=0, \\
g_{1}(2,2)=-12, & g_{1}(2,6)=0.5979, & g_{1}(2,10)=0, & g_{1}(2,12)=0, \\
g_{1}(3,2)=0, & g_{1}(3,6)=0, & g_{1}(3,10)=0, & g_{1}(3,12)=0, \\
g_{1}(4,2)=0, & g_{1}(4,6)=0, & g_{1}(4,10)=0, & g_{1}(4,12)=0, \\
g_{1}(5,2)=0, & g_{1}(5,6)=0, & g_{1}(5,10)=0, & g_{1}(5,12)=0, \\
g_{1}(6,2)=0, & g_{1}(6,6)=0, & & g_{1}(6,10)=0,  \tag{4.14}\\
g_{1}(7,2)=0, & g_{1}(7,6)=0, & g_{1}(6,12)=0.4690, \\
g_{1}(8,2)=0, & g_{1}(8,6)=0, & & g_{1}(8,10)=0, \\
g_{1}(9,2)=0, & g_{1}(9,6)=-1, & g_{1}(7,12)=0, \\
g_{1}(11,2)=0, & g_{1}(11,6)=0, & g_{1}(9,10)=-1, & g_{1}(8,12)=0, \\
g_{1}(12,2)=0, & g_{1}(12,6)=-24.5177, & g_{1}(12,10)=0, & g_{1}(9,12)=0, \\
0, & g_{1}(12,12)=-10 .
\end{array}
$$

$$
\boldsymbol{G}_{\mathbf{1}}=\left[\begin{array}{cccccccccccc}
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0  \tag{4.15}\\
0 & -12 & 0 & 0 & 0 & 0.5979 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -11 & 0 & 0 & 0 & 0 & 0 & 0.4690 \\
0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -7 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -9 & -1 \\
0 & 0 & 0 & 0 & 0 & -24.5177 & 0 & 0 & 0 & 0 & 0 & -10
\end{array}\right] .
$$

When $\boldsymbol{G}_{\mathbf{1}}$ is designed as above, the eigenvalues of $\left(\boldsymbol{A}+\boldsymbol{G}_{\boldsymbol{1}} \boldsymbol{C}\right)$ are all negative which means Hurwitz. Note that $\boldsymbol{G}_{\mathbf{1}}$ in (4.15) is not the only the matrix that satisfies the design condition. The system designer can change the values if the design conditions are not broken.

We can also design $\boldsymbol{H}_{\mathbf{1}}$ that satisfies (4.11). Since $\operatorname{ker} \boldsymbol{C}=\mathbf{0}$, we design $\boldsymbol{H}_{\mathbf{1}}$ satisfying $\operatorname{ker} \boldsymbol{H}_{\mathbf{1}}=\mathcal{S}_{1}^{*}$ as follows

$$
\boldsymbol{H}_{\mathbf{1}}=\left[\begin{array}{llllllllllll}
0 & 0.6987 & 0 & 0 & 0 & -0.0619 & 0 & 0 & 0 & 0 & 0 & 0.0145 \tag{4.16}
\end{array}\right],
$$

which is same as the first row vector of $\boldsymbol{P}_{\mathbf{1}}$. The other row vectors of $\boldsymbol{P}_{\mathbf{1}}$ also satisfy the design condition but the values do not appear in the residue $r_{1}(t)$. This means the only $\boldsymbol{H}_{\mathbf{1}}$ in (4.16) can be used. Now, with $\boldsymbol{G}_{\mathbf{1}}$ and $\boldsymbol{H}_{\mathbf{1}}$, we can make a fault detection filter of the first actuator. The process of designing fault detection filters for the remaining actuators is equivalent to the above.

Moving on to the next, we design a residue $r_{2}(t)$ that can detect the fault of the second actuator. In this case, represent the observer gain and the filter matrix are denoted by $\boldsymbol{G}_{\boldsymbol{2}}$ and $\boldsymbol{H}_{\mathbf{2}}$. Since $\boldsymbol{S}_{\mathbf{2}}$ is expressed as $\left[\boldsymbol{L}_{\mathbf{1}} \boldsymbol{L}_{\mathbf{3}} \boldsymbol{L}_{\mathbf{4}}\right]$ in (4.17), the projection matrix $\boldsymbol{P}_{\mathbf{2}}$ that satisfies $\boldsymbol{P}_{\mathbf{2}} \boldsymbol{S}_{\mathbf{2}}=\mathbf{0}$ can be derived as follows

$$
\boldsymbol{S}_{\mathbf{2}}=10^{-5} \times\left[\begin{array}{ccc}
0 & 0 & 0  \tag{4.17}\\
0.0145 & 0.0145 & 0.0145 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
-0.3277 & 0.3277 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -0.3259 \\
0 & 0 & 0 \\
0.6987 & 0.6987 & -0.6987
\end{array}\right]
$$

$$
\boldsymbol{P}_{\mathbf{2}}=\left[\begin{array}{cccccccccccc}
0 & 0.6987 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0622 & 0 & -0.0145  \tag{4.18}\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
$$

After that, we determine the elements of $\boldsymbol{G}_{\mathbf{2}}$ which is needed to satisfy the design condition in (4.19). The elements of $\boldsymbol{G}_{\mathbf{2}}$ in $i$-th row and $j$-th column are denoted by $g_{2}(i, j),(i \in$ $\mathbf{1 2}, j \in \mathbf{1 2})$. The degrees of freedom of $\boldsymbol{G}_{\mathbf{2}}$ are chosen to make $\left(\boldsymbol{A}+\boldsymbol{G}_{\mathbf{2}} \boldsymbol{C}\right)$ has no positive eigenvalues. In this condition, $\boldsymbol{G}_{\mathbf{2}}$ can be expressed as (4.20).

$$
\begin{array}{llll}
g_{2}(1,2)=-1, & g_{2}(1,6)=0, & g_{2}(1,10)=0, & g_{2}(1,12)=0 \\
g_{2}(2,2)=-9.9170, & g_{2}(2,6)=0, & g_{2}(2,10)=-0.8835, & g_{2}(2,12)=0.2060 \\
g_{2}(3,2)=0, & g_{2}(3,6)=0, & g_{2}(3,10)=0, & g_{2}(3,12)=0 \\
g_{2}(4,2)=0, & g_{2}(4,6)=0, & g_{2}(4,10)=0, & g_{2}(4,12)=0 \\
g_{2}(5,2)=0, & g_{2}(5,6)=-1, & g_{2}(5,10)=0, & g_{2}(5,12)=0 \\
g_{2}(7,2)=0, & g_{2}(7,6)=0, & g_{2}(7,10)=0, & g_{2}(7,12)=0 \\
g_{2}(8,2)=0, & g_{2}(8,6)=0, & g_{2}(8,10)=0, & g_{2}(8,12)=0 \\
g_{2}(9,2)=0, & g_{2}(9,6)=0, & g_{2}(9,10)=-1, & g_{2}(9,12)=0 \\
g_{2}(10,2)=47.2570, & g_{2}(10,6)=0, & g_{2}(10,10)=4.2099, & g_{2}(10,12)=-0.9816, \\
g_{2}(11,2)=0, & g_{2}(11,6)=0, & g_{2}(11,10)=0, & g_{2}(11,12)=-1 \\
g_{2}(12,2)=0.2060, & g_{2}(12,6)=0, & g_{2}(12,10)=0.0184, & g_{2}(12,12)=-0.0043 .
\end{array}
$$

$$
\boldsymbol{G}_{\mathbf{2}}=\left[\begin{array}{cccccccccccc}
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0  \tag{4.20}\\
0 & -9.9170 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.8835 & 0 & 0.2060 \\
0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -7 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & -1 & 0 & 0 \\
0 & 47.2570 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4.2099 & 0 & -0.9816 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -9 & -1 \\
0 & 0.2060 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0184 & 0 & -0.0043
\end{array}\right] .
$$

In this case, all eigenvalues of $\left(\boldsymbol{A}+\boldsymbol{G}_{\mathbf{2}} \boldsymbol{C}\right)$ are not positive but some of the eigenvalues are zero which means marginally stable.

We can also design $\boldsymbol{H}_{\mathbf{2}}$ that satisfies (4.11). Since $\operatorname{ker} \boldsymbol{C}=\mathbf{0}$, we design $\boldsymbol{H}_{\mathbf{2}}$ satisfying $\operatorname{ker} \boldsymbol{H}_{\mathbf{2}}=\mathcal{S}_{2}^{*}$ as follows

$$
\boldsymbol{H}_{\mathbf{2}}=\left[\begin{array}{llllllllllll}
0 & 0.6987 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0622 & 0 & -0.0145 \tag{4.21}
\end{array}\right]
$$

which is same as the first row vector of $\boldsymbol{P}_{\mathbf{2}}$. The other row vectors of $\boldsymbol{P}_{\mathbf{2}}$ also satisfy the design condition but the values do not appear in the residue $r_{2}(t)$. This means the only $\boldsymbol{H}_{\mathbf{2}}$ in (4.21) can be used.

Now, let us design a residue $r_{3}(t)$ that can detect the fault of the third actuator this time. The observer gain and the filter matrix are denoted by $\boldsymbol{G}_{\mathbf{3}}$ and $\boldsymbol{H}_{\mathbf{3}}$. Since $\boldsymbol{S}_{\mathbf{3}}$ is expressed as $\left[\begin{array}{lll}\boldsymbol{L}_{\mathbf{1}} & \boldsymbol{L}_{\mathbf{2}} & \boldsymbol{L}_{\mathbf{4}}\end{array}\right]$ in (4.22), the projection matrix $\boldsymbol{P}_{\mathbf{3}}$ that satisfies $\boldsymbol{P}_{\mathbf{3}} \boldsymbol{S}_{\mathbf{3}}=\mathbf{0}$ can be derived as follows

$$
\begin{align*}
& \boldsymbol{S}_{\mathbf{3}}=10^{-5} \times\left[\begin{array}{ccc}
0 & 0 & 0 \\
0.0145 & 0.0145 & 0.0145 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
-0.3277 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0.3259 & -0.3259 \\
0 & 0 & 0 \\
0.6987 & -0.6987 & -0.6987
\end{array}\right],  \tag{4.22}\\
& \boldsymbol{P}_{\mathbf{3}}=\left[\begin{array}{cccccccccccc}
0 & 0.6987 & 0 & 0 & 0 & 0.0619 & 0 & 0 & 0 & 0 & 0 & 0.0145 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right] . \tag{4.23}
\end{align*}
$$

And then, we select the elements of $\boldsymbol{G}_{\mathbf{3}}$ which is needed to satisfy the design condition in (4.24). The elements of $\boldsymbol{G}_{\boldsymbol{3}}$ in $i$-th row and $j$-th column are denoted by $g_{3}(i, j),(i \in$ $\mathbf{1 2}, j \in \mathbf{1 2}$ ). The degrees of freedom of $\boldsymbol{G}_{\mathbf{3}}$ are chosen to make $\left(\boldsymbol{A}+\boldsymbol{G}_{\mathbf{3}} \boldsymbol{C}\right)$ has no positive eigenvalues. Then, $\boldsymbol{G}_{\boldsymbol{3}}$ can be expressed as (4.25).

$$
\begin{array}{lllll}
g_{3}(1,2) & =-1, & g_{3}(1,6)=0, & & g_{3}(1,10)=0, \\
g_{3}(2,2)=-9.9170, & g_{3}(2,6)=0, & g_{3}(2,10)=-0.8835, & & \left.g_{3}(1,12)=0,12\right)=0.2060, \\
g_{3}(3,2)=0, & g_{3}(3,6)=0, & g_{3}(3,10)=0, & & g_{3}(3,12)=0, \\
g_{3}(4,2)=0, & g_{3}(4,6)=0, & g_{3}(4,10)=0, & & g_{3}(4,12)=0, \\
g_{3}(5,2)=0, & g_{3}(5,6)=-1, & g_{3}(5,10)=0, & & g_{3}(5,12)=0, \\
g_{3}(6,2)=47.2570, & g_{3}(6,6)=0, & g_{3}(6,10)=4.2099, & g_{3}(6,12)=-0.9816, \\
g_{3}(7,2)=0, & g_{3}(7,6)=0, & g_{3}(7,10)=0, & g_{3}(7,12)=0, \\
g_{3}(8,2)=0, & g_{3}(8,6)=0, & g_{3}(8,10)=0, & g_{3}(8,12)=0, \\
g_{3}(9,2)=0, & g_{3}(9,6)=0, & g_{3}(9,10)=-1, & g_{3}(9,12)=0, \\
g_{3}(11,2)=0, & g_{3}(11,6)=0, & g_{3}(11,10)=0, & g_{3}(11,12)=-1, \\
g_{3}(12,2)=0.2060, & g_{3}(12,6)=0, & g_{3}(12,10)=0.0184, & g_{3}(12,12)=-0.0043 .
\end{array}
$$

$$
\boldsymbol{G}_{\mathbf{3}}=\left[\begin{array}{cccccccccccc}
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0  \tag{4.25}\\
0 & -9.9179 & 0 & 0 & 0 & -0.8785 & 0 & 0 & 0 & 0 & 0 & -0.2060 \\
0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.8785 & 0 & 0 & 0 & -0.0778 & 0 & 0 & 0 & 0 & 0 & -0.0182 \\
0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -7 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -9 & -1 \\
0 & -0.2060 & 0 & 0 & 0 & -0.0182 & 0 & 0 & 0 & 0 & 0 & -0.0043
\end{array}\right] .
$$

In (4.25), all eigenvalues of $\left(\boldsymbol{A}+\boldsymbol{G}_{\mathbf{3}} \boldsymbol{C}\right)$ are not positive but some of the eigenvalues are zero which means marginally stable.

We can also design $\boldsymbol{H}_{\mathbf{3}}$ that satisfies (4.11). Since $\operatorname{ker} \boldsymbol{C}=\mathbf{0}$, we design $\boldsymbol{H}_{\mathbf{3}}$ satisfying $\operatorname{ker} \boldsymbol{H}_{3}=\mathcal{S}_{3}^{*}$ as follows

$$
\boldsymbol{H}_{\mathbf{3}}=\left[\begin{array}{llllllllllll}
0 & 0.6987 & 0 & 0 & 0 & 0.0619 & 0 & 0 & 0 & 0 & 0 & 0.0145 \tag{4.26}
\end{array}\right],
$$

which is same as the first row vector of $\boldsymbol{P}_{\mathbf{3}}$. The other row vectors of $\boldsymbol{P}_{\mathbf{3}}$ also satisfy the design condition but the values do not appear in the residue $r_{3}(t)$. This means the only $\boldsymbol{H}_{3}$ in (4.26) can be used.

Lastly, we design a residue $r_{4}(t)$ that can only detect the fault of the fourth actuator. The observer gain and the filter matrix are denoted by $\boldsymbol{G}_{\mathbf{4}}$ and $\boldsymbol{H}_{\mathbf{4}}$. Since $\boldsymbol{S}_{\mathbf{4}}$ is expressed as $\left[\boldsymbol{L}_{\mathbf{1}} \boldsymbol{L}_{\mathbf{2}} \boldsymbol{L}_{\mathbf{3}}\right]$ in (4.27), the projection matrix $\boldsymbol{P}_{\mathbf{4}}$ that satisfies $\boldsymbol{P}_{\mathbf{4}} \boldsymbol{S}_{\mathbf{4}}=\mathbf{0}$ can be derived as follows

$$
\boldsymbol{S}_{\mathbf{4}}=10^{-5} \times\left[\begin{array}{ccc}
0 & 0 & 0  \tag{4.27}\\
0.0145 & 0.0145 & 0.0145 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
-0.3277 & 0 & 0.3277 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0.3259 & 0 \\
0 & 0 & 0 \\
0.6987 & -0.6987 & 0.6987
\end{array}\right],
$$

$$
\boldsymbol{P}_{4}=\left[\begin{array}{cccccccccccc}
0 & 0.6987 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.0622 & 0 & -0.0145  \tag{4.28}\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
$$

Then, we select the elements of $\boldsymbol{G}_{\boldsymbol{4}}$ which is needed to satisfy the design condition in (4.29). The elements of $\boldsymbol{G}_{\mathbf{4}}$ in $i$-th row and $j$-th column are denoted by $g_{4}(i, j),(i \in \mathbf{1 2}, j \in \mathbf{1 2})$. The degrees of freedom of $\boldsymbol{G}_{\boldsymbol{4}}$ are chosen to make $\left(\boldsymbol{A}+\boldsymbol{G}_{\boldsymbol{4}} \boldsymbol{C}\right)$ has no positive eigenvalues. Then, $\boldsymbol{G}_{\boldsymbol{4}}$ can be expressed as (4.30).

$$
\begin{array}{llll}
g_{4}(1,2)=-1, & g_{4}(1,6)=0, & g_{4}(1,10)=0, & g_{4}(1,12)=0 \\
g_{4}(2,2)=-0.0992, & g_{4}(2,6)=0, & g_{4}(2,10)=0.0088, & g_{4}(2,12)=0.0020, \\
g_{4}(3,2)=0, & g_{4}(3,6)=0, & g_{4}(3,10)=0, & g_{4}(3,12)=0 \\
g_{4}(4,2)=0, & g_{4}(4,6)=0, & g_{4}(4,10)=0, & g_{4}(4,12)=0 \\
g_{4}(5,2)=0, & g_{4}(5,6)=-1, & g_{4}(5,10)=0, & g_{4}(5,12)=0 \\
g_{4}(6,2)=0, & g_{4}(6,6)=0, & g_{4}(6,10)=0, & g_{4}(6,12)=0 \\
g_{4}(7,2)=0, & g_{4}(7,6)=0, & g_{4}(7,10)=0, & g_{4}(7,12)=0 \\
g_{4}(8,2)=0, & g_{4}(8,6)=0, & g_{4}(8,10)=0, & g_{4}(8,12)=0 \\
g_{4}(9,2)=0, & g_{4}(9,6)=0, & g_{4}(9,10)=-1, & g_{4}(9,12)=0 \\
g_{4}(10,2)=112.2619, & g_{4}(10,6)=0, & g_{4}(10,10)=-10.0007, & g_{4}(10,12)=-2.3320 \\
g_{4}(11,2)=0, & g_{4}(11,6)=0, & g_{4}(11,10)=0, & g_{4}(11,12)=-1 \\
g_{4}(12,2)=0.0020, & g_{4}(12,6)=0, & g_{4}(12,10)=-0.0002, & g_{4}(12,12)=0
\end{array}
$$

$$
\boldsymbol{G}_{\mathbf{4}}=\left[\begin{array}{cccccccccccc}
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0  \tag{4.30}\\
0 & -0.0992 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0088 & 0 & 0.0020 \\
0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -7 & -1 & 0 & 0 \\
0 & 112.2619 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -10.0007 & 0 & -2.3320 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & -1 \\
0 & 0.0020 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.0002 & 0 & 0
\end{array}\right] .
$$

In (4.30), all eigenvalues of $\left(\boldsymbol{A}+\boldsymbol{G}_{\boldsymbol{4}} \boldsymbol{C}\right)$ are not positive but some of the eigenvalues are zero which means marginally stable.

We can also design $\boldsymbol{H}_{4}$ that satisfies (4.11). Since $\operatorname{ker} \boldsymbol{C}=\mathbf{0}$, we design $\boldsymbol{H}_{4}$ satisfying ker $\boldsymbol{H}_{4}=\mathcal{S}_{4}^{*}$ as follows

$$
\boldsymbol{H}_{4}=\left[\begin{array}{llllllllllll}
0 & 0.6987 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.0622 & 0 & -0.0145 \tag{4.31}
\end{array}\right],
$$

which is same as the first row vector of $\boldsymbol{P}_{\mathbf{4}}$. The other row vectors of $\boldsymbol{P}_{\mathbf{4}}$ also satisfy the design condition but the values do not appear in the residue $r_{4}(t)$. This means the only $\boldsymbol{H}_{4}$ in (4.31) can be used.

### 4.4 Designing fault detection filters in detailed quadrotor model

As we did in the previous section, we find the observer gain $\boldsymbol{G}_{\boldsymbol{i}}$ and the filter matrix $\boldsymbol{H}_{\boldsymbol{i}}$ in the linear state equation of detailed quadrotor model. To distinguish previously designed matrices, $\boldsymbol{G}_{\boldsymbol{i}}$ and $\boldsymbol{H}_{\boldsymbol{i}}$ start with $\boldsymbol{G}_{\mathbf{5}}$ and $\boldsymbol{H}_{\mathbf{5}}$.

Matrices $\boldsymbol{G}_{\mathbf{5}}$ and $\boldsymbol{H}_{\mathbf{5}}$ are to build the residue $r_{5}(t)$ that can detect only the first actuator fault signal in the detailed model. From the linearization, $\boldsymbol{S}_{5}$ still remained as
$\left[\begin{array}{lll}\boldsymbol{L}_{\mathbf{2}} & \boldsymbol{L}_{\mathbf{3}} & \boldsymbol{L}_{4}\end{array}\right]$. So, the projection matrix $\boldsymbol{P}_{\mathbf{5}}$ is same as $\boldsymbol{P}_{\mathbf{1}}$. When we expressed the value in $i$-th row and $j$-th column of $\boldsymbol{G}_{\mathbf{5}}$ as $g_{5}(i, j),(i \in \mathbf{1 2}, j \in \mathbf{1 2})$, the elements of $\boldsymbol{G}_{\mathbf{5}}$ that are needed to satisfy the design condition are denoted in (4.32).

$$
\begin{align*}
g_{5}(1,2)=-1, & g_{5}(1,6)=0, & g_{5}(1,10)=0, & g_{5}(1,12)=0 \\
g_{5}(2,2)=-3, & g_{5}(2,6)=0, & g_{5}(2,10)=0, & g_{5}(2,12)=0, \\
g_{5}(3,2)=0, & g_{5}(3,6)=0, & g_{5}(3,10)=0, & g_{5}(3,12)=0, \\
g_{5}(4,2)=0, & g_{5}(4,6)=0, & g_{5}(4,10)=0, & g_{5}(4,12)=0, \\
g_{5}(5,2)=0, & g_{5}(5,6)=-1, & g_{5}(5,10)=0, & g_{5}(5,12)=0, \\
g_{5}(6,2)=0, & g_{5}(6,6)=-3, & g_{5}(6,10)=0, & g_{5}(6,12)=-0.2170, \\
g_{5}(7,2)=0, & g_{5}(7,6)=0, & g_{5}(7,10)=0, & g_{5}(7,12)=0, \\
g_{5}(8,2)=0, & g_{5}(8,6)=0, & g_{5}(8,10)=0, & g_{5}(8,12)=0, \\
g_{5}(9,2)=0, & g_{5}(9,6)=0, & g_{5}(9,10)=-1, & g_{5}(9,12)=0, \\
g_{5}(11,2)=0, & g_{5}(11,6)=0, & g_{5}(11,10)=0, & g_{5}(11,12)=-1,  \tag{4.32}\\
g_{5}(12,2)=0, & g_{5}(12,6)=-3.7520, & g_{5}(12,10)=0, & g_{5}(12,12)=-3 .
\end{align*}
$$

The elements whose values are not determined in (4.32) mean the degrees of freedom of $\boldsymbol{G}_{\mathbf{5}}$. These degrees of freedom are used to make that $\left(\boldsymbol{A}+\boldsymbol{G}_{\mathbf{5}} \boldsymbol{C}\right)$ is Hurwitz.

$$
\boldsymbol{G}_{\mathbf{5}}=\left[\begin{array}{cccccccccccc}
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0  \tag{4.33}\\
0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & -0.2170 \\
0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -7 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & -1 \\
0 & 0 & 0 & 0 & 0 & -3.7520 & 0 & 0 & 0 & 0 & 0 & -3
\end{array}\right] .
$$

Finally, $\boldsymbol{G}_{\mathbf{5}}$ with all elements are determined is expressed in (4.33). Since $\boldsymbol{S}_{\mathbf{5}}$ is same as $\boldsymbol{S}_{\mathbf{1}}$, the filter matrix $\boldsymbol{H}_{\mathbf{5}}$ is also same as $\boldsymbol{H}_{\mathbf{1}}$ due to (4.11).

Let us move on to the step of designing the residue $r_{6}(t)$, which is only for the second actuator fault in the detailed model. Since $\boldsymbol{S}_{\mathbf{6}}$ still remained as $\left[\boldsymbol{L}_{\mathbf{1}} \boldsymbol{L}_{\mathbf{3}} \boldsymbol{L}_{4}\right.$ ], the projection matrix $\boldsymbol{P}_{\mathbf{6}}$ is same as $\boldsymbol{P}_{\mathbf{2}}$. When we expressed the value in $i$-th row and $j$-th column of $\boldsymbol{G}_{\mathbf{6}}$ as $g_{6}(i, j),(i \in \mathbf{1 2}, j \in \mathbf{1 2})$, the elements of $\boldsymbol{G}_{\mathbf{6}}$ that are needed to satisfy the design condition are denoted in (4.34).

$$
\begin{align*}
g_{6}(1,2) & =-1, & g_{6}(1,6)=0, & g_{6}(1,10)=0, \\
g_{6}(2,2)=-3, & g_{6}(2,6)=0, & g_{6}(2,10)=0, & g_{6}(1,12)=0, \\
g_{6}(3,2)=0, & g_{6}(3,6)=0, & g_{6}(3,10)=0, & g_{6}(3,12)=0, \\
g_{6}(4,2)=0, & g_{6}(4,6)=0, & g_{6}(4,10)=0, & g_{6}(4,12)=0, \\
g_{6}(5,2)=0, & g_{6}(5,6)=-1, & g_{6}(5,10)=0, & g_{6}(5,12)=0, \\
g_{6}(7,2)=0, & g_{6}(7,6)=0, & g_{6}(7,10)=0, & g_{6}(7,12)=0, \\
g_{6}(8,2)=0, & g_{6}(8,6)=0, & g_{6}(8,10)=0, & g_{6}(8,12)=0, \\
g_{6}(9,2)=0, & g_{6}(9,6)=0, & g_{6}(9,10)=-1, & g_{6}(9,12)=0, \\
g_{6}(10,2)=0, & g_{6}(10,6)=-0.8749, & g_{6}(10,10)=-2.1306, & g_{6}(10,12)=-0.2158, \\
g_{6}(11,2)=0, & g_{6}(11,6)=0, & g_{6}(11,10)=0, & g_{6}(11,12)=-1, \\
g_{6}(12,2)=0, & g_{6}(12,6)=-3.7520, & g_{6}(12,10)=0, & g_{6}(12,12)=-3 .
\end{align*}
$$

The elements whose values are not determined in (4.34) mean the degrees of freedom of $\boldsymbol{G}_{\boldsymbol{6}}$. These degrees of freedom are used to make that $\left(\boldsymbol{A}+\boldsymbol{G}_{\boldsymbol{6}} \boldsymbol{C}\right)$ is Hurwitz. Finally, $\boldsymbol{G}_{\boldsymbol{6}}$ with all elements are determined is expressed in (4.35).

$$
\boldsymbol{G}_{\mathbf{6}}=\left[\begin{array}{cccccccccccc}
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0  \tag{4.35}\\
0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & -0.2170 \\
0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -7 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.8749 & 0 & 0 & 0 & -2.1306 & 0 & -0.2158 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & -1 \\
0 & 0 & 0 & 0 & 0 & -3.7520 & 0 & 0 & 0 & 0 & 0 & -3
\end{array}\right] .
$$

Since $\boldsymbol{S}_{\mathbf{6}}$ is same as $\boldsymbol{S}_{\mathbf{2}}$, the filter matrix $\boldsymbol{H}_{\mathbf{6}}$ is also same as $\boldsymbol{H}_{\mathbf{2}}$ due to (4.11).

Next, let us design $r_{7}(t)$, which is only for the third actuator fault in the detailed model. Since $\boldsymbol{S}_{\boldsymbol{7}}$ still remained as $\left[\boldsymbol{L}_{\mathbf{1}} \boldsymbol{L}_{\mathbf{2}} \boldsymbol{L}_{\mathbf{4}}\right.$ ], the projection matrix $\boldsymbol{P}_{\boldsymbol{7}}$ is same as $\boldsymbol{P}_{\mathbf{3}}$. When we expressed the value in $i$-th row and $j$-th column of $\boldsymbol{G}_{\boldsymbol{7}}$ as $g_{\boldsymbol{7}}(i, j),(i \in \mathbf{1 2}, j \in \mathbf{1 2})$, the elements of $\boldsymbol{G}_{\boldsymbol{7}}$ that are needed to satisfy the design condition are denoted in (4.36).

$$
\begin{array}{rlrlrl}
g_{7}(1,2) & =-1, & g_{7}(1,6)=0, & g_{7}(1,10)=0, & g_{7}(1,12)=0, \\
g_{7}(2,2) & =0, & g_{7}(2,6)=0, & g_{7}(2,10)=0, & g_{7}(2,12)=0, \\
g_{7}(3,2) & =0, & g_{7}(3,6)=0, & g_{7}(3,10)=0, & g_{7}(3,12)=0, \\
g_{7}(4,2) & =0, & g_{7}(4,6)=0, & g_{7}(4,10)=0, & g_{7}(4,12)=0, \\
g_{7}(5,2)=0, & g_{7}(5,6)=-1, & g_{7}(5,10)=0, & g_{7}(5,12)=0, \\
g_{7}(6,2)=-10.4480, & g_{7}(6,6)=0, & g_{7}(6,10)=0, & g_{7}(6,12)=0, \\
g_{7}(7,2)=0, & g_{7}(7,6)=0, & g_{7}(7,10)=0, & g_{7}(7,12)=0, \\
g_{7}(8,2)=0, & g_{7}(8,6)=0, & g_{7}(8,10)=0, & g_{7}(8,12)=0,  \tag{4.36}\\
g_{7}(9,2)=0, & g_{7}(9,6)=0, & g_{7}(9,10)=-1, & g_{7}(9,12)=0, \\
g_{7}(11,2)=0, & g_{7}(11,6)=0, & g_{7}(11,10)=0, & g_{7}(11,12)=-1, \\
g_{7}(12,2)=0, & g_{7}(12,6)=-0.1939, & g_{7}(12,10)=0, & g_{7}(12,12)=0 .
\end{array}
$$

The elements whose values are not determined in (4.36) mean the degrees of freedom of $\boldsymbol{G}_{\boldsymbol{7}}$. These degrees of freedom are used to make that $\left(\boldsymbol{A}+\boldsymbol{G}_{\boldsymbol{7}} \boldsymbol{C}\right)$ is Hurwitz. Finally, $\boldsymbol{G}_{\boldsymbol{7}}$ with all elements are determined is expressed in (4.37).

$$
\boldsymbol{G}_{\boldsymbol{7}}=\left[\begin{array}{cccccccccccc}
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0  \tag{4.37}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -10.4480 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -7 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & -1 \\
0 & 0 & 0 & 0 & 0 & -0.1939 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] .
$$

Since $\boldsymbol{S}_{\boldsymbol{7}}$ is same as $\boldsymbol{S}_{\mathbf{3}}$, the filter matrix $\boldsymbol{H}_{\boldsymbol{7}}$ is also same as $\boldsymbol{H}_{\mathbf{3}}$ due to (4.11).

The last, let us design $r_{8}(t)$, which is only for the fourth actuator fault in the detailed model. Since $\boldsymbol{S}_{\mathbf{8}}$ still remained as $\left[\boldsymbol{L}_{\mathbf{1}} \boldsymbol{L}_{\mathbf{2}} \boldsymbol{L}_{\mathbf{3}}\right]$, the projection matrix $\boldsymbol{P}_{\mathbf{8}}$ is same as $\boldsymbol{P}_{\mathbf{4}}$. When we expressed the value in $i$-th row and $j$-th column of $\boldsymbol{G}_{\mathbf{8}}$ as $g_{8}(i, j),(i \in \mathbf{1 2}, j \in$ 12), the elements of $\boldsymbol{G}_{\mathbf{8}}$ that are needed to satisfy the design condition are denoted in (4.38).

$$
\begin{align*}
g_{8}(1,2)=-1, & g_{8}(1,6)=0, & g_{8}(1,10)=0, & g_{8}(1,12)=0, \\
g_{8}(2,2)=-0.9254, & g_{8}(2,6)=0, & g_{8}(2,10)=0, & g_{8}(2,12)=0, \\
g_{8}(3,2)=0, & g_{8}(3,6)=0, & g_{8}(3,10)=0, & g_{8}(3,12)=0, \\
g_{8}(4,2)=0, & g_{8}(4,6)=0, & g_{8}(4,10)=0, & g_{8}(4,12)=0, \\
g_{8}(5,2)=0, & g_{8}(5,6)=-1, & g_{8}(5,10)=0, & g_{8}(5,12)=0, \\
g_{8}(7,2)=0, & g_{8}(7,6)=0, & g_{8}(7,10)=0, & g_{8}(7,12)=0, \\
g_{8}(8,2)=0, & g_{8}(8,6)=0, & g_{8}(8,10)=0, & g_{8}(8,12)=0, \\
g_{8}(9,2)=0, & g_{8}(9,6)=0, & g_{8}(9,10)=-1, & g_{8}(9,12)=0, \\
g_{8}(10,2)=-1, & g_{8}(10,6)=0, & g_{8}(10,10)=0, & g_{8}(10,12)=0, \\
g_{8}(11,2)=0, & g_{8}(11,6)=0, & g_{8}(11,10)=0, & g_{8}(11,12)=-1, \\
g_{8}(12,2)=4.2886, & g_{8}(12,6)=0, & g_{8}(12,10)=-0.2402, & g_{8}(12,12)=0
\end{align*}
$$

The elements whose values are not determined in (4.38) mean the degrees of freedom of $\boldsymbol{G}_{\mathbf{8}}$. These degrees of freedom are used to make that $\left(\boldsymbol{A}+\boldsymbol{G}_{\mathbf{8}} \boldsymbol{C}\right)$ is Hurwitz. Finally, $\boldsymbol{G}_{\mathbf{8}}$ with all elements are determined is expressed in (4.39).

$$
\boldsymbol{G}_{\mathbf{8}}=\left[\begin{array}{cccccccccccc}
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0  \tag{4.39}\\
0 & -0.9254 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -7 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -9 & -1 \\
0 & 4.2886 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.2402 & 0 & 0
\end{array}\right] .
$$

Since $\boldsymbol{S}_{\mathbf{8}}$ is same as $\boldsymbol{S}_{\mathbf{4}}$, the filter matrix $\boldsymbol{H}_{\mathbf{8}}$ is also same as $\boldsymbol{H}_{\mathbf{4}}$ due to (4.11).

## Simulation

### 5.1 Simulation environment

The whole simulation environment is implemented through MATLAB Simulink. In this simulation, two different quadrotor dynamics introduced in Chapter 3, LQR controller, and four fault detection filters are implemented.

We set the operating point of the quadrotor at $\boldsymbol{x}^{*}=\left[\begin{array}{llllllllll}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{array} 0\right.$ which means the quadrotor is hovering at a height of 1 m in the $e_{3}$ direction. All positive constants in the quadrotor model that we used are obtained in [17]. The total simulation time is 60 seconds, and we assume that one particular actuator has a fault signal when 20 seconds after the simulation starts. Also, we define a situation in which one actuator has a $20 \%$ RPM power degradation as a fault.

### 5.2 Simulation result

The actuator fault detection results in linear state equation of the classical quadrotor model are shown in Figure 5.1, Figure 5.2, Figure 5.3, and Figure 5.4. In Figure 5.1, the simulation result shows the reaction of each residue when the fault occurs only in first actuator. Only the residue $r_{1}(t)$ shows a response to the fault signal and the rest of the residues are not affected by the fault and keep remaining as zero.

In Figure 5.2, the simulation result shows the reaction of each residue when the fault occurs only in second actuator. Only the residue $r_{2}(t)$ shows a response to the fault signal and the rest of the residues are not affected by the fault and keep remaining as zero.

In Figure 5.3, the simulation result shows the reaction of each residue when the fault occurs only in third actuator. Only the residue $r_{3}(t)$ shows a response to the fault signal and the rest of the residues are not affected by the fault and keep remaining as zero.

In Figure 5.4, the simulation result shows the reaction of each residue when the fault occurs only in fourth actuator. Only the residue $r_{4}(t)$ shows a response to the fault signal and the rest of the residues are not affected by the fault and keep remaining as zero. These results indicate that the residues $r_{1}(t), r_{2}(t), r_{3}(t)$, and $r_{4}(t)$ are well designed as expected.

Also, we apply these residues $r_{1}(t), r_{2}(t), r_{3}(t)$, and $r_{4}(t)$ to the linear state equation of detailed quadrotor model. If these residues are work well, the detailed quadrotor model will not necessary for fault detection. The simulation results are shown from Figure 5.5 to Figure 5.8. In this case, we can see that all residues react to all fault effects. For example, residue $r_{1}(t)$ has a signal not only for the first actuator fault but also for the second, third, and fourth actuator fault. In this case, the residue signal shown by the rest of the actuators is small. However, the size of the residue signal depends on the size of the fault, so if the fault occurs in another actuator with different sizes, it is hard to tell which actuator fails. Figure 5.9 shows the example of this situation. The left side of Figure 5.9 shows the residue $r_{1}(t)$ when the first actuator has a $5 \%$ RPM power degradation. The right side of Figure 5.9 shows the residue $r_{1}(t)$ when the fourth actuator stops the operating. As you can see, when we only see $r_{1}(t)$, we may judge that the first actuator has a fault. But, on the right side of Figure 5.9, it is actually caused by a fault of the fourth actuator. To prevent this misjudgment, we need to apply the actual dynamics of the quadrotor in the model.

Now, The actuator fault detection results in linear state equation of the detailed quadrotor model are shown in Figure 5.10, Figure 5.11, Figure 5.12, and Figure 5.13. In Figure 5.10, only the residue $r_{5}(t)$ shows a response to the fault signal and the rest of the residues are not affected by the fault and keep remaining as zero.

In Figure 5.11, only the residue $r_{6}(t)$ shows a response to the fault signal and the rest of the residues are not affected by the fault and keep remaining as zero.

In Figure 5.12, only the residue $r_{7}(t)$ shows a response to the fault signal and the rest of the residues are not affected by the fault and keep remaining as zero.

In Figure 5.13, only the residue $r_{8}(t)$ shows a response to the fault signal and the rest
of the residues are not affected by the fault and keep remaining as zero.
Finally, we apply our fault detection filters to nonlinear quadrotor model described in (3.1). Though our fault detection filters are suitable for the detailed linear model, Figure 5.14, Figure 5.15, Figure 5.16, and Figure 5.17 show that fault detection is possible under the limited condition when the quadrotor is hovering on the nonlinear model.


Figure 5.1: Residues when the first actuator fault occurs in the classical quadrotor model.


Figure 5.2: Residues when the second actuator fault occurs in the classical quadrotor model.


Figure 5.3: Residues when the third actuator fault occurs in the classical quadrotor model.


Figure 5.4: Residues when the fourth actuator fault occurs in the classical quadrotor model.


Figure 5.5: Model changes from classical to the detailed model in Figure 5.1.


Figure 5.6: Model changes from classical to the detailed model in Figure 5.2.


Figure 5.7: Model changes from classical to the detailed model in Figure 5.3.


Figure 5.8: Model changes from classical to the detailed model in Figure 5.4.

(a) residue $r_{1}(t)$ when first actuator has a $5 \% \mathrm{RPM}(\mathrm{b})$ residue $r_{1}(t)$ when the fourth actuator stops the oppower degradation erating

Figure 5.9: Comparison of $r_{1}(t)$ in two faulty case


Figure 5.10: Residues when the first actuator fault occurs in the detailed quadrotor model.


Figure 5.11: Residues when the second actuator fault occurs in the detailed quadrotor model.


Figure 5.12: Residues when the third actuator fault occurs in the detailed quadrotor model.


Figure 5.13: Residues when the fourth actuator fault occurs in the detailed quadrotor model.


Figure 5.14: Residues when the first actuator fault occurs in the nonlinear quadrotor model.


Figure 5.15: Residues when the second actuator fault occurs in the nonlinear quadrotor model.


Figure 5.16: Residues when the third actuator fault occurs in the nonlinear quadrotor model.


Figure 5.17: Residues when the fourth actuator fault occurs in the nonlinear quadrotor model.

## Conclusion

In this thesis, to obtain the effect of aerodynamics induced by propellers on actuator fault detection performance, we used two different quadrotor dynamic models. One is the classical model that presents the quadrotor dynamics in an ideal environment. The other is the detailed model that includes the aerodynamic effect of the propellers. We linearized each of the two linear models to apply the fault detection method.

We use the geometric approach as a fault detection method that uses characteristics of subspaces in observable LTI system. If we can find these subspaces with special algorithms from the system, we can easily design fault detection filters by using the subspaces that we obtained. In this thesis, the whole process of the algorithms and filter design are introduced and we actually design eight fault detection filters with specific design factors.

To verify the performance of the designed fault detection filters, MATLAB Simulink was used as a simulator. We confirm the performance of the fault detection filter suitable for the classical quadrotor model. Also, we show the effect of aerodynamics has affected fault detection performance, which means fault detection is not possible. And then, we suggest new fault detection filters suitable for the quadrotor model with the aerodynamic effect of propellers. We also confirm that our fault detection filters are valid in the nonlinear model when the quadrotor is hovering.

However, the fault detection filter designed in this thesis is only usable when the quadrotor is hovering. Since the actual quadrotor moves in an outdoor environment, we should consider fault detection in the full nonlinear system that takes into account various situations. This will be left as future work.

## A

## Matrices in linearization

Here, matrices in (3.21) are fully expressed. We have two linear models, one is the classical quadrotor model with $\boldsymbol{f}_{\boldsymbol{D}}=\mathbf{0}, \boldsymbol{\tau}_{\boldsymbol{D}}=\mathbf{0}$, and $T_{i}=k_{w} w_{i}{ }^{2}$ in (3.1) and (3.3). The other is the detailed quadrotor model using BEMT. To prevent confusion, the matrices in the linear state equation of the classical model are expressed by $\boldsymbol{A}_{\boldsymbol{c}}$ and $\boldsymbol{B}_{\boldsymbol{c}}$ as follows

$$
\begin{align*}
\dot{\boldsymbol{x}} & =\boldsymbol{A}_{c} \boldsymbol{x}+\boldsymbol{B}_{c} \boldsymbol{u},  \tag{A.1}\\
\boldsymbol{y} & =\boldsymbol{C} \boldsymbol{x} .
\end{align*}
$$

First, the matrices $\boldsymbol{A}_{\boldsymbol{c}}$ and $\boldsymbol{B}_{\boldsymbol{c}}$ are given by (A.2) and (A.3), respectively, where $J_{2}=1 / I_{1}$, $J_{4}=1 / I_{2}, J_{6}=1 / I_{3}$, and $I=\operatorname{diag}\left(I_{1}, I_{2}, I_{3}\right)$. And then, the matrices $\boldsymbol{A}_{\mathbf{0}}$ and $\boldsymbol{B}_{\mathbf{0}}$ in the linear state equation of the detailed model in (3.21) is derived in (A.4) . Note that calculation of the matrices in (3.21) are not the contribution of this thesis. All the matrices in (3.21) are calculated in [17].

We do not describe what $\overline{\boldsymbol{A}}$ and $\overline{\boldsymbol{B}}$ are here because we do not consider the effect of external wind $\boldsymbol{W}$. Also in this thesis, $\boldsymbol{B}_{\mathbf{0}}$ is equal to $\boldsymbol{B}_{\boldsymbol{c}}$ because both linear model deal with a same hovering situation. Only $\boldsymbol{A}_{\mathbf{0}}=\operatorname{diag}\left(\boldsymbol{A}_{\mathbf{0}, \mathbf{1}}, \boldsymbol{A}_{\mathbf{0}, \mathbf{2}}, \boldsymbol{A}_{\mathbf{0}, \mathbf{3}} \boldsymbol{A}_{\mathbf{0}, \mathbf{4}}\right)$, where each sub-matrix $\boldsymbol{A}_{0, i}$ is defined by (A.4).

$$
\boldsymbol{A}_{\boldsymbol{c}}=\left[\begin{array}{cccccccccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0  \tag{A.2}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & g & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -g & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & g & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] .
$$

$$
\boldsymbol{B}_{\boldsymbol{c}}=\left[\begin{array}{cccc}
0 & 0 & 0 & 0  \tag{A.3}\\
k_{w} / m & k_{w} / m & k_{w} / m & k_{w} / m \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-k_{w} J_{4} l & 0 & k_{w} J_{4} l & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & k_{w} J_{2} l & 0 & -k_{w} J_{2} l \\
0 & 0 & 0 & 0 \\
J_{6} k_{m} k_{w} & -J_{6} k_{m} k_{w} & J_{6} k_{m} k_{w} & -J_{6} k_{m} k_{w}
\end{array}\right] .
$$

$$
\begin{aligned}
& \boldsymbol{A}_{\mathbf{0 , 1}}=\left[\begin{array}{ll}
0 & 1 \\
0 & -4 k_{z} w^{*} / m
\end{array}\right], \\
& \boldsymbol{A}_{\mathbf{0 , 2}}=\left[\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & -4 c k_{w}\left(w^{*}\right)^{2} / m & 4 k_{w}\left(w^{*}\right)^{2} / m & 0 \\
0 & 0 & 0 & 1 \\
0 & -4 k_{w} J_{4} c d\left(w^{*}\right)^{2} & 0 & -2 k_{z} J_{4} l^{2} w^{*}
\end{array}\right], \\
& \boldsymbol{A}_{\mathbf{0 , 3}}=\left[\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & -4 c k_{w}\left(w^{*}\right)^{2} / m & -4 k_{w}\left(w^{*}\right)^{2} / m & 0 \\
0 & 0 & 0 & 1 \\
0 & 4 k_{w} J_{2} c d\left(w^{*}\right)^{2} & 0 & -2 k_{z} J_{2} l^{2} w^{*}
\end{array}\right], \\
& \boldsymbol{A}_{\mathbf{0 , 4}}=\left[\begin{array}{ll}
0 & 1 \\
0 & -4 k_{w} J_{6} c l^{2}\left(w^{*}\right)^{2}
\end{array}\right] .
\end{aligned}
$$

## General case when $\boldsymbol{C}$ is not full rank with TE-PCS

Here, detailed process of UOSA and CAISA in geometric approach will be introduced. We use Tennessee-Eastman process control system (TE-PCS) which is a typical industrial process with four actuator inputs. In TE-PCS, the matrix $\boldsymbol{C}$ is not full rank, so we can explain the whole process of UOSA and CAISA. From [25], we can obtain LTI system as follows:

$$
\begin{align*}
& \dot{\boldsymbol{x}}(\boldsymbol{t})=\boldsymbol{A} \boldsymbol{x}(\boldsymbol{t})+\boldsymbol{B} \boldsymbol{u}(\boldsymbol{t})+\sum_{i=1}^{4} \boldsymbol{L}_{i} m_{i}(t)  \tag{B.1}\\
& \boldsymbol{y}(\boldsymbol{t})=\boldsymbol{C} \boldsymbol{x}(\boldsymbol{t})
\end{align*}
$$

$$
\boldsymbol{A}=\left[\begin{array}{cccccccc}
-1.333 & 0 & 0 & 0 & 0 & 0 & 0 & 0  \tag{B.2}\\
0 & -11 & -2.5 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.5 & -4.1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -20.1 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{array}\right]
$$

$$
\begin{gather*}
\boldsymbol{B}=\left[\begin{array}{cccc}
2 & 0 & 0 & 0 \\
0 & 0 & 0 & 8 \\
0 & 0 & 0 & 0 \\
4.5 & 0 & -1.125 & 0 \\
12.75 & 0 & -0.75 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{array}\right],  \tag{B.3}\\
\boldsymbol{C}=\left[\begin{array}{ccccccc}
1.333 & -4.25 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 8 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.075 & 1.5 \\
0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1
\end{array}\right] . \tag{B.4}
\end{gather*}
$$

Consider that we design full order observer in (4.2), then we can obtain the error system (4.4) with four residues. To design four fault detection filters, we have to calculate the infimal $(\boldsymbol{C}, \boldsymbol{A})$-invariant subspace $\mathcal{W}_{i}^{*}$ and infimal unobservable subspace $\mathcal{S}_{i}^{*}$ through CAISA and UOSA.

First, we will show how to obtain $(\boldsymbol{C}, \boldsymbol{A})$-invariant subspace $\mathcal{W}_{1}^{*}$ through CAISA.
In Theorem 4.1, when $k=0$, as the initial condition is $\boldsymbol{W}_{\mathbf{1}}^{\mathbf{0}}=\mathbf{0}$, we can notice the projection matrix $\boldsymbol{P}_{\boldsymbol{W}, \mathbf{1}}^{0}$ is the nonzero arbitrary matrix which satisfies $\boldsymbol{P}_{\boldsymbol{W}, \mathbf{1}}^{0} \boldsymbol{W}_{\mathbf{1}}^{\mathbf{0}}=\mathbf{0}$. Then, $\boldsymbol{T}_{W, 1}^{0}$, which satisfies following condition for all possible $\boldsymbol{P}_{W, 1}^{0}$ is a zero matrix.

$$
\left[\begin{array}{c}
P_{W, 1}^{0}  \tag{B.5}\\
C
\end{array}\right] T_{W, 1}^{0}=0
$$

Therefore, in $\boldsymbol{W}_{\mathbf{1}}^{\mathbf{1}}=\left[\begin{array}{llll}\boldsymbol{L}_{\mathbf{2}} & \boldsymbol{L}_{\mathbf{3}} & \boldsymbol{L}_{\mathbf{4}} & \boldsymbol{A} \boldsymbol{T}_{\boldsymbol{W}, \mathbf{1}}^{\mathbf{0}}\end{array}\right]$, the rank of $\boldsymbol{W}_{\mathbf{1}}^{\mathbf{1}}$ is 3 .
Now, move on the next step, when $k=1$, we can obtain projection matrix $\boldsymbol{P}_{\boldsymbol{W}, \mathbf{1}}^{1}$ that satisfies $\boldsymbol{P}_{\boldsymbol{W}, \mathbf{1}}^{\mathbf{1}} \boldsymbol{W}_{1}^{\mathbf{1}}=\mathbf{0}$ as follows:

$$
\boldsymbol{P}_{W, 1}^{\mathbf{1}}=\left[\begin{array}{cccccccc}
1 & 1 & 1 & 0.75 & -1.125 & 0 & 1 & -8  \tag{B.6}\\
1 & 1 & -1 & 0.75 & -1.125 & 0 & 1 & -8 \\
1 & 1 & 1 & 0.75 & -1.125 & 0 & -1 & -8 \\
1 & 0 & 1 & 0.75 & -1.125 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & -8
\end{array}\right]
$$

Then, $\boldsymbol{T}_{\boldsymbol{W}, \mathbf{1}}^{1}$, which satisfies following condition for all possible $\boldsymbol{P}_{\boldsymbol{W}, \mathbf{1}}^{0}$ is a zero matrix because the rank of $\left[\boldsymbol{P}_{\boldsymbol{W}, \mathbf{1}}^{\mathbf{1}}, \boldsymbol{C}\right]^{T}$ is 8 . Therefore, in $\boldsymbol{W}_{\mathbf{1}}^{\mathbf{2}}=\left[\boldsymbol{L}_{\mathbf{2}} \boldsymbol{L}_{\mathbf{3}} \boldsymbol{L}_{\mathbf{4}} \boldsymbol{A} \boldsymbol{T}_{\boldsymbol{W}, \mathbf{1}}^{\mathbf{1}}\right]$, the rank of $\boldsymbol{W}_{\mathbf{1}}^{\mathbf{2}}$ is 3 . Now, CAISA is stopped due to Rank $\boldsymbol{W}_{\mathbf{1}}^{\mathbf{2}}=\operatorname{Rank} \boldsymbol{W}_{\mathbf{1}}^{\mathbf{1}}$.

Then, $(\boldsymbol{C}, \boldsymbol{A})$-invariant subspace $\mathcal{W}_{1}^{*}$ is the image of $\boldsymbol{W}_{\mathbf{1}}^{\mathbf{1}}$, i.e., $\mathcal{W}_{1}^{*}=\operatorname{Im} \boldsymbol{W}_{\mathbf{1}}^{\mathbf{1}}=$ $\operatorname{Im}\left[\begin{array}{lll}\boldsymbol{L}_{\mathbf{2}} & \boldsymbol{L}_{\mathbf{3}} & \boldsymbol{L}_{4}\end{array}\right]$.

With $\mathcal{W}_{1}^{*}$, we can obtain unobservable subspace $\mathcal{S}_{1}^{*}$ through UOSA. In Theorem 4.2, when $k=0$, as the initial condition is $\boldsymbol{S}_{\mathbf{1}}^{\mathbf{0}}=\boldsymbol{I}_{\mathbf{8} \times \mathbf{8}}$, we can notice the projection matrix $\boldsymbol{P}_{\boldsymbol{S}, \mathbf{1}}^{\mathbf{0}}$ is the zero matrix which satisfies $\boldsymbol{P}_{\boldsymbol{S}, \mathbf{1}}^{\mathbf{0}} \boldsymbol{S}_{\mathbf{1}}^{\mathbf{0}}=\mathbf{0}$. Then, $\boldsymbol{T}_{\boldsymbol{S}, \mathbf{1}}^{\mathbf{0}}$, which satisfies the following condition can be expressed by four linearly independent column vectors.

$$
\begin{gather*}
{\left[\begin{array}{c}
\boldsymbol{P}_{S, 1}^{\mathbf{0}} \boldsymbol{A} \\
\boldsymbol{C}
\end{array}\right] \boldsymbol{T}_{S, 1}^{\mathbf{0}}=\mathbf{0},}  \tag{B.7}\\
\boldsymbol{T}_{S, \mathbf{1}}^{\mathbf{0}}=\left[\begin{array}{cccc}
4.25 & 0 & 0 & 0 \\
1.333 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
1.5 & 0 & 0 & 1.5 \\
0.075 & 0 & 0 & 0.075 \\
0 & 0 & 0 & 0
\end{array}\right] . \tag{B.8}
\end{gather*}
$$

Therefore, in $\boldsymbol{S}_{1}^{1}=\left[\boldsymbol{W}_{1}^{1} \boldsymbol{T}_{S, 1}^{0}\right]=\left[\begin{array}{llll}\boldsymbol{L}_{2} & \boldsymbol{L}_{3} & \boldsymbol{L}_{4} & \boldsymbol{T}_{S, 1}^{0}\end{array}\right]$, the rank of $\boldsymbol{S}_{1}^{1}$ is 7 .
Move on the next step, when $k=1$, we can obtain projection matrix $\boldsymbol{P}_{S, 1}^{1}$ that satisfies $\boldsymbol{P}_{S, 1}^{1} S_{1}^{1}=\mathbf{0}$ as follows:

$$
\boldsymbol{P}_{S, 1}^{1}=\left[\begin{array}{llllllll}
-0.3136 & 1 & 0 & 0 & 0 & 0 & 0 & -8 \tag{B.9}
\end{array}\right] .
$$

Then, $\boldsymbol{T}_{S, \mathbf{1}}^{1}$, which satisfies the following condition can be expressed by three linearly independent column vectors.

$$
\begin{gather*}
{\left[\begin{array}{c}
\boldsymbol{P}_{S, \mathbf{1}}^{\mathbf{1}} \boldsymbol{A} \\
\boldsymbol{C}
\end{array}\right] \boldsymbol{T}_{\boldsymbol{S}, \mathbf{1}}^{\mathbf{1}}=\mathbf{0},}  \tag{B.10}\\
\boldsymbol{T}_{\boldsymbol{S , 1}}^{\mathbf{1}}=\left[\begin{array}{ccc}
4.25 & 0 & 0 \\
1.333 & 0 & 0 \\
-5.15443 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1.5 \\
0 & 0 & 0.075 \\
0 & 0 & 0
\end{array}\right] . \tag{B.11}
\end{gather*}
$$

Therefore, in $\boldsymbol{S}_{1}^{2}=\left[\boldsymbol{W}_{1}^{1} \boldsymbol{T}_{S, 1}^{1}\right]=\left[\begin{array}{llll}\boldsymbol{L}_{2} & \boldsymbol{L}_{3} & \boldsymbol{L}_{4} & \boldsymbol{T}_{S, 1}^{1}\end{array}\right]$, the rank of $\boldsymbol{S}_{\mathbf{1}}^{2}$ is 6 .
Move on the next step, when $k=2$, we can obtain projection matrix $\boldsymbol{P}_{S, 1}^{2}$ that satisfies $P_{S, 1}^{2} S_{1}^{2}=\mathbf{0}$ as follows:

$$
\boldsymbol{P}_{S, 1}^{2}=\left[\begin{array}{cccccccc}
-0.3136 & 1 & 0 & 0 & 0 & 0 & 0 & -8  \tag{B.12}\\
0 & 1 & 0.2586 & 0 & 0 & 0 & 0 & -8
\end{array}\right] .
$$

Then, $\boldsymbol{T}_{S, 1}^{2}$, which satisfies the following condition can be expressed by two linearly independent column vectors.

$$
\left[\begin{array}{c}
P_{S, 1}^{2} A  \tag{B.13}\\
C
\end{array}\right] T_{S, 1}^{2}=0
$$

$$
\boldsymbol{T}_{S, 1}^{\mathbf{2}}=\left[\begin{array}{cc}
0 & 0  \tag{B.14}\\
0 & 0 \\
0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 1.5 \\
0 & 0.075 \\
0 & 0
\end{array}\right]
$$

Therefore, in $\boldsymbol{S}_{1}^{\mathbf{3}}=\left[\boldsymbol{W}_{1}^{\mathbf{1}} \boldsymbol{T}_{S, 1}^{2}\right]=\left[\begin{array}{llll}\boldsymbol{L}_{2} & \boldsymbol{L}_{3} & \boldsymbol{L}_{4} & \boldsymbol{T}_{S, 1}^{2}\end{array}\right]$, the rank of $\boldsymbol{S}_{1}^{\mathbf{3}}$ is 5 .
Move on the next step, when $k=3$, we can obtain projection matrix $\boldsymbol{P}_{S, 1}^{3}$ that satisfies $\boldsymbol{P}_{S, 1}^{3} S_{1}^{3}=\mathbf{0}$ as follows:

$$
\boldsymbol{P}_{S, 1}^{3}=\left[\begin{array}{cccccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & -8  \tag{B.15}\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

Then, $\boldsymbol{T}_{S, 1}^{3}$, which satisfies the following condition can be expressed by two linearly independent column vectors.

$$
\begin{gather*}
{\left[\begin{array}{c}
\boldsymbol{P}_{S, 1}^{3} \boldsymbol{A} \\
\boldsymbol{C}
\end{array}\right] \boldsymbol{T}_{\boldsymbol{S}, \mathbf{1}}^{3}=\mathbf{0},}  \tag{B.16}\\
\boldsymbol{T}_{S, \mathbf{1}}^{\mathbf{3}}=\left[\begin{array}{cc}
0 & 0 \\
0 & 0 \\
0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 1.5 \\
0 & 0.075 \\
0 & 0
\end{array}\right] . \tag{B.17}
\end{gather*}
$$

Therefore, in $\boldsymbol{S}_{1}^{4}=\left[\boldsymbol{W}_{1}^{1} \boldsymbol{T}_{S, 1}^{3}\right]=\left[\begin{array}{llll}\boldsymbol{L}_{2} & \boldsymbol{L}_{3} & \boldsymbol{L}_{4} & \boldsymbol{T}_{S, 1}^{3}\end{array}\right]$, the rank of $\boldsymbol{S}_{1}^{4}$ is 5. Now, UOSA is stopped due to Rank $\boldsymbol{S}_{1}^{4}=\operatorname{Rank} \boldsymbol{S}_{1}^{3}$.

Then, unobservable subspace $\mathcal{S}_{1}^{*}$ is the image of $\boldsymbol{S}_{1}^{3}$, i.e., $\mathcal{S}_{1}^{*}=\operatorname{Im} \boldsymbol{S}_{1}^{3}=\operatorname{Im}\left[\boldsymbol{L}_{2} \boldsymbol{L}_{3} \boldsymbol{L}_{4} \boldsymbol{T}_{\boldsymbol{S}, \mathbf{1}}^{2}\right]$.
In the same process, we can obtain the unobservable subspaces $\mathcal{S}_{2}^{*}, \mathcal{S}_{3}^{*}$, and $\mathcal{S}_{4}^{*}$ that are needed to design each fault detection filters. In this thesis, we do not describe all processes but the calculation process is the same as above.

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## 요약문

## 프로필러의 공기역학적 특성을 고려한 ㅋㅝㅝ드로터 액추에이터 고장 감지

운행중인 쿼드로터에 발생할 수 있는 오동작의 원인 중 하나로는 액추에이터에 발생하는 고장이 있다. 이때 고장 신호는 모터의 출력 저하 또는 완전 정지의 형태로 나타난다. 이러 한 고장은 운행중인 쿼드로터에게 추락과 같은 치명적인 결과를 야기할 수 있으므로, 고장 발생 시 이를 감지하고 대처할 수 있도록 하는 조치가 필요하다. 이를 위해서는 먼저, 어느 액추에이터에 고장이 발생했는지 찾아내는 과정이 필요하다.

이에 본 연구는 쿼드로터의 동역학 모델을 기반으로 액추에이터에 발생하는 고장 신 호를 감지하는 방법에 대해 서술한다. 본 연구에서 사용한 쿼드로터의 동역학 모델은 두 가지이다. 하나는 이상적인 환경에서의 동작을 가정한 모델이며, 쿼드로터의 움직임을 비 교적 간단하게 표현하므로 퀀드로터를 이용한 대부분의 연구에서 보편적으로 사용된다. 다른 하나는 프로펠러의 회전에 의해 발생하는 공기역학적 특성이 쿼드로터의 동역학에 미치는 영향을 고려한 모델이다. 본 연구에서는 고장 감지 기법을 적용하기 위해 두 가지 선형 모델 각각에 대해 쿼드로터가 제자리에서 비행하고 있는 상황에서의 선형화 모델을 수립하였다.

본 연구에서 사용한 고장 감지 기법은 선형 모델의 상태 추정 오차로 표현되는 오차 시스템의 선형부분공간을 이용한 기하학적 접근법이며, 선형화 모델과 오차 시스템의 관 측불가능한 선형부분공간의 특성을 통해 액추에이터 고장을 감지하는 데 적합한 필터들을 설계하였다.

설계한 고장 감지 필터의 성능을 분석하기 위해, MATLAB Simulink 기반의 시뮬레이 션을 사용하였다. 우선 이상적인 쿼드로터 선형 모델에 적합한 고장 감지 필터들을 시뮬 레이션으로 구현하고 성능을 확인하였다. 그리고, 이 필터들을 공기역학적 특성이 고려된 쿼드로터 선형 모델에 적용하여, 프로펠러의 회전에 의한 공기역학적 특성이 고장 감지 성 능에 미치는 영향을 확인하였다. 이후 공기역학적 특성이 고려된 쿼드로터 모델에 적합한 새로운 고장 감지 필터들을 시뮬레이션으로 구현하고 성능을 확인하여 실제 쿼드로터 운행 환경에 적합한 액추에이터 고장 감지 필터를 제시하였다.
주요어휘: 고장 감지, 기하학적 접근법, 관측불가능한 선형부분공간, 쿼드로터, 액추에이터, 공기역학적 특성


[^0]:    ${ }^{1}$ Declaration of Ethical Conduct in Research: I, as a graduate student of DGIST, hereby declare that I have not committed any acts that may damage the credibility of my research. These include, but are not limited to: falsification, thesis written by someone else, distortion of research findings or plagiarism. I affirm that my thesis contains honest conclusions based on my own careful research under the guidance of my thesis advisor.

