



The boundary of Rauzy fractal and discrete tilings

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ABSTRACT

We present two methods of constructing the Rauzy fractal by partitioning all points within it into disjoint sets, refer to as layers. We show how these layered structures of the Rauzy fractal can plot the boundary of the fractal effectively. By generalizing the self-replicating pattern of this structure, we demonstrate a new way of discrete tilings of two-dimensional plane.

1. Introduction

1.1. A brief introduction to the Rauzy fractal

Let a **letter** be an integer $0, 1, 2, \dots$ and a **word** be a string of letters. For example, the following is a word consists of three letters $0, 1, 2$:

$$[w_3] = [0102010]. \quad (1)$$

The **Rauzy substitution** is a rule that transforms a word into another word by substituting each letter as follows:

$$0 \rightarrow [01], \quad 1 \rightarrow [02], \quad 2 \rightarrow [0]. \quad (2)$$

Continuing the substitution, we can generate a sequence of words whose length grows indefinitely. For example, successive application of the Rauzy substitution to the word $[0]$ yields the following sequence of words:

$$[w_0] = [0], \quad (3)$$

$$[w_1] = [01], \quad (4)$$

$$[w_2] = [0102], \quad (5)$$

$$[w_3] = [0102010], \quad (6)$$

$$[w_4] = [0102010010201]. \quad (7)$$

A **prefix** of a word is a substring that appears at the beginning of the word. One of the characteristics of the Rauzy substitution is that the prefixes remain unchanged. The prefix of the words $[w_n]$ with large index n is as follows:

$$[0102010010201010201001020102010010201010 \dots]. \quad (8)$$

[1] proposed a construction of a two dimensional region, called the **Rauzy fractal**, in the following way. First, let us assign three standard basis vectors to the letters $0, 1, 2$ as follows.

$$0 \rightarrow e_0 = (1, 0, 0), \quad (9)$$

$$1 \rightarrow e_1 = (0, 1, 0), \quad (10)$$

$$2 \rightarrow e_2 = (0, 0, 1). \quad (11)$$

Each word is then assigned to a three dimensional vector, called the **word vector** by summing up the basis vectors corresponding to the letters in the word. For example, the first seven prefixes of the word $[w_3]$ in (1) are associated with the following vectors:

$$[0] \leftrightarrow (1, 0, 0), \quad (12)$$

$$[01] \leftrightarrow (1, 1, 0), \quad (13)$$

$$[010] \leftrightarrow (2, 1, 0), \quad (14)$$

$$[0102] \leftrightarrow (2, 1, 1), \quad (15)$$

$$[01020] \leftrightarrow (3, 2, 1), \quad (16)$$

$$[010201] \leftrightarrow (3, 3, 1), \quad (17)$$

$$[0102010] \leftrightarrow (4, 3, 1). \quad (18)$$

As the lengths of prefixes of the word (8) increases, the corresponding word vector converges asymptotically to the vector v_∞ :

$$v_\infty \approx (0.850, 0.462, 0.252). \quad (19)$$

The plane P perpendicular to v_∞ is called the **contracting plane**. All word vectors corresponding to the prefixes of the word $[w_n]$ can be

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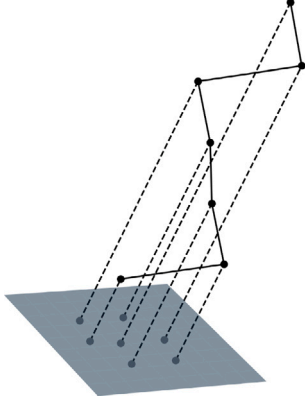


Fig. 1. The illustration of the construction of the Rauzy fractal by projection.

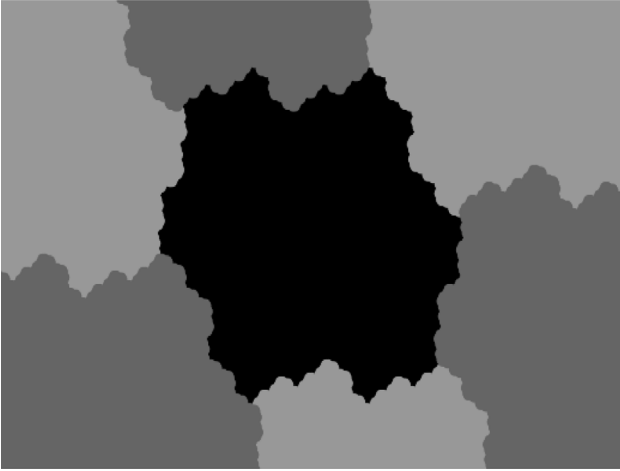


Fig. 2. The discrete tiling of two-dimensional plane by the copies of the Rauzy fractal.

projected onto the plane P , as shown in Fig. 1. [1] found that these projected points cluster within a compact region in the plane P , known as the Rauzy fractal.

There are several interesting properties of the Rauzy fractal. First, as its name indicates, the boundary of the Rauzy fractal is fractal-like. As Fig. 2 illustrates, copies of the Rauzy fractal can be arranged to tile two-dimensional plane without gaps or overlaps. Second, the Rauzy fractal can be split into a collection of smaller Rauzy fractals. Fig. 3 shows a subdivision of the Rauzy fractal into three regions, labeled 1, 2, and 4. The labels indicate the number of copies of the Rauzy fractal that fits into each region. The region labeled 4 (light gray) is the four times the size of the region labeled 1, and twice the size of the region labeled 2.

1.2. Pisot substitutions and their domains

A **substitution on d letters** is a mapping that transforms a word into another word by replacing each character with a predefined sequence of d letters. Below are examples of substitutions on three letters.

$$\sigma_0 : 0 \rightarrow [01], \quad 1 \rightarrow [02], \quad 2 \rightarrow [0], \quad (20)$$

$$\sigma_1 : 0 \rightarrow [12], \quad 1 \rightarrow [2], \quad 2 \rightarrow [0], \quad (21)$$

$$\sigma_2 : 0 \rightarrow [0102], \quad 1 \rightarrow [2], \quad 2 \rightarrow [0], \quad (22)$$

$$\sigma_3 : 0 \rightarrow [01], \quad 1 \rightarrow [2], \quad 2 \rightarrow [0]. \quad (23)$$

Let $[\mathbf{w}_n]$ be the sequence generated by a substitution σ from the initial word $[\mathbf{w}_0] = [0]$. Let $\mathbf{v}_n$ denote the word vector of $[\mathbf{w}_n]$. Then, we can

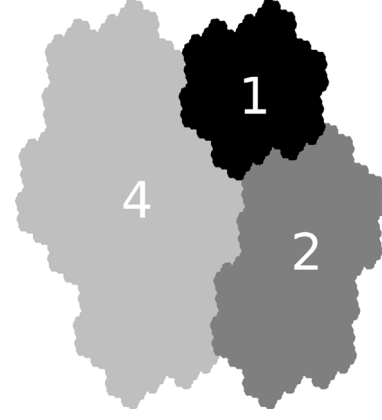


Fig. 3. The subdivision of the Rauzy fractal into smaller regions.

Table 1

Pisot substitutions, Pisot matrices, and Pisot numbers.

Pisot substitution	Pisot matrix	Characteristic polynomial	Pisot number
σ_0	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	$1 + \lambda + \lambda^2 = \lambda^3$	1.8393
σ_1	$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$	$1 + \lambda = \lambda^3$	1.3247
σ_2	$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$	$1 + \lambda + 2\lambda^2 = \lambda^3$	2.5468
σ_3	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$	$1 + \lambda^2 = \lambda^3$	1.4656

associate a unique $d \times d$ matrix M that satisfies the following equation for all $n \geq 0$:

$$\mathbf{v}_{n+1} = M\mathbf{v}_n \quad (24)$$

If the matrix M has a unique real eigenvalue $\lambda > 1$ and all other (possibly complex) eigenvalues have absolute values less than 1, it is called a **Pisot matrix**. In this case, the number λ is referred to as a **Pisot number**, and the associated substitution σ is known as a **Pisot substitution**. All substitutions in Eqs. (20)–(23) are Pisot substitutions. Table 1 presents the Pisot matrices, their characteristic polynomials, and the corresponding Pisot numbers for each substitution.

For a 3×3 Pisot matrix M , let $\mathbf{v}_\infty$ be the eigenvector of M corresponding to the Pisot number λ . Let P be the **contracting plane**, defined as the hyperplane orthogonal to $\mathbf{v}_\infty$. Now, let $[\mathbf{w}_n]$ be the sequence of words generated by the Pisot substitution σ corresponding to M , with the initial word $[\mathbf{w}_0] = [0]$. Let $\mathbf{v}$ be a prefix of $[\mathbf{w}_n]$ for sufficiently large n , and let $\pi(\mathbf{v})$ denote the projection of $\mathbf{v}$ onto the contracting plane P .

To represent $\pi(\mathbf{v})$ as a two-dimensional point, we can choose two orthonormal basis vectors, $\mathbf{r}_0$ and $\mathbf{r}_1$, on P and express the coordinates as follows. Such a basis can be obtained by applying the Gram–Schmidt process to the set $\{\mathbf{v}_\infty, \mathbf{e}_0, \mathbf{e}_1\}$.

Let $[\mathbf{w}_n]$ be the sequence of words generated by the Pisot substitution σ corresponding to M , with the initial word $[\mathbf{w}_0] = [0]$. Let $\mathbf{v}$ be a prefix of $[\mathbf{w}_n]$ for sufficiently large n , and let $\pi(\mathbf{v})$ denote the projection of $\mathbf{v}$ onto the contracting plane P . To represent $\pi(\mathbf{v})$ as a two dimensional point, we can choose two orthonormal basis vectors, $\mathbf{r}_0$ and $\mathbf{r}_1$, on P , which can be obtained by applying the Gram–Schmidt process to the set $\{\mathbf{v}_\infty, \mathbf{e}_0, \mathbf{e}_1\}$, and express the coordinates as follows:

$$\pi(\mathbf{v}) = (\mathbf{r}_1 \cdot \mathbf{v}, \mathbf{r}_2 \cdot \mathbf{v}). \quad (25)$$

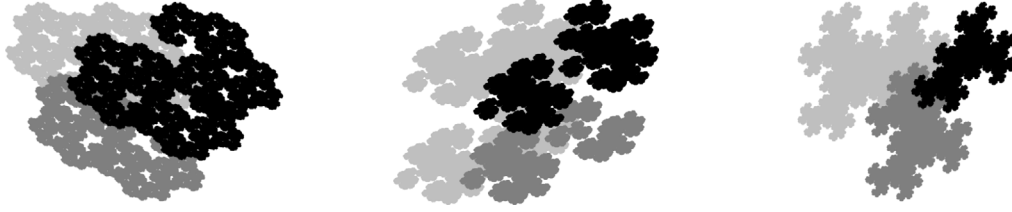


Fig. 4. The Pisot domain of the substitutions σ_1 , σ_2 , and σ_3 in three colors.

Let R_n be the set of all points $\pi(v)$ for all prefixes v of w_n . The **Pisot domain** R is then defined as the union of all R_n .

$$R = \bigcup_{n \geq 0} R_n \quad (26)$$

By plotting the point $\pi(v)$ and assigning colors based on the last letter of the prefix v , a fractal-like tiling of the domain is obtained. For example, images in Fig. 4 show the Pisot domains for the substitutions σ_1 , σ_2 , σ_3 from Eqs. (21)–(23), respectively, each represented in three colors. The Pisot domain of σ_0 is previously displayed in Fig. 1.

1.3. Historical background and related works

[2,3] reveals how Rauzy substitution plays a key role in understanding the substitutive systems in symbolic dynamics. The Rauzy substitution is an example of a Pisot substitution, and the relationship between Pisot substitutions and the Pisot numbers is studied in [4,5]. Recent studies by [6–8] provide valuable insights into the fractal domains in one, two, and three dimensional spaces.

The tessellation of the two-dimensional plane by the Rauzy fractal generalizes to *Pisot conjecture*, which states that every Pisot substitution on d letters can discretely tile the $d - 1$ dimensional space. The Pisot conjecture has been extensively studied in [9–12].

The Pisot conjecture further generalizes to the pure *discrete spectrum conjecture*, which states that every dynamical system defined by a Pisot substitution on d letters has a pure discrete spectrum. This conjecture has been proven only for $d \leq 2$; see [11,13–15] for more details.

The fractal structure of the Rauzy fractal is concentrated along its boundary. In Rauzy's original construction, the boundary of the Rauzy fractal is obtained by plotting a sufficiently large number of points that consists the *interior*. The resolution of the boundary can be improved only by increasing the number of plotted points; however, this approach does not allow for the exclusion of interior points from the boundary. Therefore, this method is computationally expensive, and poses challenges in achieving a high-resolution representation of the boundary. Although various alternative methods for constructing the Rauzy fractal have been explored in [16–19], the challenge of the high computational cost associated with accurately constructing its boundary remains.

1.4. The self-replicating pattern of the Rauzy fractal

Let us observe two words $[w_3]$ and $[w_6]$ in the sequence of words generated by the Rauzy substitution.

$$[w_3] = [0102010] \quad (27)$$

$$\begin{aligned} [w_6] &= [0102010 | 010201 | 0102010 | \\ &\quad \underbrace{0 \text{ trim}} \quad \underbrace{1 \text{ trim}} \quad \underbrace{0 \text{ trim}} \\ &\quad \underbrace{0102} \quad \underbrace{0102010} \quad \underbrace{010201} \quad \underbrace{0102010} \\ &\quad \underbrace{3 \text{ trims}} \quad \underbrace{0 \text{ trim}} \quad \underbrace{1 \text{ trim}} \quad \underbrace{0 \text{ trim}} \\ &= [w_6^{(0)} w_6^{(1)} w_6^{(2)} w_6^{(3)} w_6^{(4)} w_6^{(5)} w_6^{(6)}] \end{aligned} \quad (28)$$

We can see that the word $[w_6]$ is formed by concatenating the words $[w_6^{(i)}]$, for $0 \leq i \leq 5$, separated by bars, where each is obtained by trimming the last 0, 1 or 3 letters from the word $[w_3]$. The sequence

of the numbers of trimmed letters is 0, 1, 0, 3, 0, 1, 0. With the exception of the letter 2, this sequence coincides with the letters in the word $[w_3]$. Thus, we will refer this sequence as the **pseudo self-replicating pattern** of the word $[0102010]$.

Let us apply the same pattern of trimming to the word $[w_6]$; this time, the trimming units are the tokenized words from $[w_6^{(0)}]$ to $[w_6^{(6)}]$. We then obtain exactly the same word $[w_9]$ as follows:

$$\begin{aligned} [w_9] &= [\underbrace{w_6^{(0)} w_6^{(1)} w_6^{(2)} w_6^{(3)} w_6^{(4)} w_6^{(5)} w_6^{(6)}}_{0 \text{ trim}} | \underbrace{w_6^{(0)} w_6^{(1)} w_6^{(2)} w_6^{(3)} w_6^{(4)} w_6^{(5)}}_{1 \text{ trim}} | \\ &\quad \underbrace{w_6^{(0)} w_6^{(1)} w_6^{(2)} w_6^{(3)} w_6^{(4)} w_6^{(5)} w_6^{(6)}}_{0 \text{ trim}} | \\ &\quad \underbrace{w_6^{(0)} w_6^{(1)} w_6^{(2)} w_6^{(3)}}_{3 \text{ trims}} | \underbrace{w_6^{(0)} w_6^{(1)} w_6^{(2)} w_6^{(3)} w_6^{(4)} w_6^{(5)} w_6^{(6)}}_{0 \text{ trim}} | \\ &\quad \underbrace{w_6^{(0)} w_6^{(1)} w_6^{(2)} w_6^{(3)} w_6^{(4)} w_6^{(5)}}_{1 \text{ trim}} | \\ &\quad \underbrace{w_6^{(0)} w_6^{(1)} w_6^{(2)} w_6^{(3)} w_6^{(4)} w_6^{(5)} w_6^{(6)}}_{0 \text{ trim}}] \end{aligned} \quad (29)$$

$$\begin{aligned} &= [01020100102010102010010201020 \\ &\quad 10010201010201001020100102010 \\ &\quad 1020100102010201001020101020100 \\ &\quad 1020101020100102010201001020 \\ &\quad 10102010010201001020101020100102] \end{aligned} \quad (30)$$

One of the key findings of this paper is that the prefixes of the word generated by the pseudo self-replicating pattern of the word $[0102010]$ are the same as the words generated by the Rauzy substitution, thus producing the Rauzy fractal. Moreover, we found that complete self-replicating patterns of certain words also produce compact domains that discretely tile two-dimensional plane.

For example, let us consider the self-replicating pattern of the word $[0120]$. Starting from the word $[s_1]$, the next word $[s_{n+1}]$ is generated by sequentially trimming the last 0, 1, 2, and 0 tokens from the words in $[s_n]$, with each token separated by a bar, as follows:

$$[s_1] = [0 | 1 | 2 | 0] \quad (31)$$

$$[s_2] = [\underbrace{0120}_{0 \text{ trim}} | \underbrace{012}_{1 \text{ trim}} | \underbrace{01}_{2 \text{ trims}} | \underbrace{0120}_{0 \text{ trim}}] \quad (32)$$

$$\begin{aligned} [s_3] &= [\underbrace{0120012010120}_{0 \text{ trim}} | \underbrace{012001201}_{1 \text{ trim}} | \\ &\quad \underbrace{0120012}_{2 \text{ trims}} | \underbrace{0120012010120}_{0 \text{ trim}}] \end{aligned} \quad (33)$$

We observe that the prefixes of the words $[s_n]$ are also prefixes of the subsequent word $[s_{n+1}]$. These prefixes can be associated with word vectors using the basis vectors e_0, e_1, e_2 , as defined in Eqs. (9)–(11). If these word vectors converge asymptotically to a vector v_∞ , we can project them onto the contracting plane, which is orthogonal to v_∞ . The projected points cluster into a compact domain, as illustrated in Fig. 5. Moreover, Fig. 6 shows that this compact domain can discretely tile the two-dimensional plane.



Fig. 5. The compact domain produced by the self-replicating pattern of [0 1 2 0].

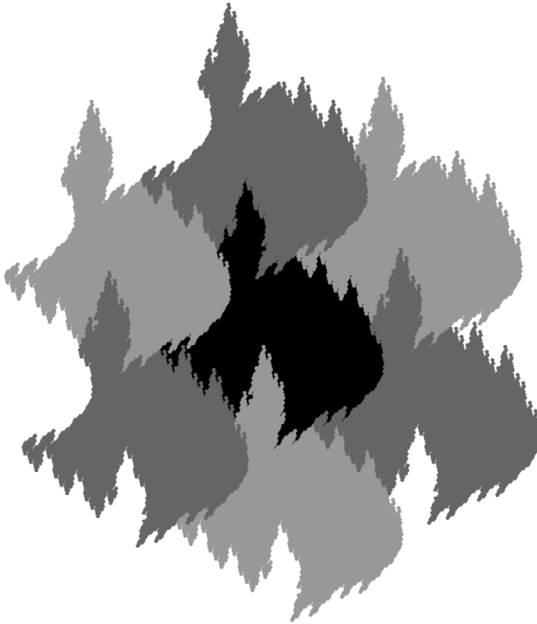


Fig. 6. The discrete tiling of the compact domain for the self-replicating pattern of [0 1 2 0].

1.5. Outline of the paper

In this paper, we propose two methods for constructing the boundary of the Rauzy fractal. Both methods are based on the idea of classifying points within the Rauzy fractal into disjoint *layers*. These layers are distinguished by their *levels*, and each point in a given layer generates its *offspring* points in the next layer. By selecting only the offspring points that lie on the boundary of the Rauzy fractal, we can effectively construct its boundary.

In Section 2, we introduce the first method for constructing the Rauzy fractal using the layered structure of the region, referred to as the A-layer. We demonstrate that each A-layer is disjoint and that the Rauzy fractal can be expressed as the disjoint union of all A-layers.

Additionally, we present an effective approach for constructing the boundary of the Rauzy fractal and discuss its computational advantages and limitations.

In Section 3, we introduce the second method for constructing the Rauzy fractal using the B-layered structure. We prove that the pseudo self-replicating pattern described in Eqs. (28) and (29) generates the same words as the Rauzy substitution. The boundary of the Rauzy fractal can be constructed similarly to the A-layered structure but using a different set of layers.

In Section 4, we define a general concept of self-replicating patterns of words and present our findings on the discrete tiling of the two-dimensional plane by sequences of integers that follow such patterns. We provide examples of self-replicating patterns and illustrate their corresponding domains. Additionally, we discuss the limitations of our formulation and explore potential generalizations.

2. A-layered structure of the Rauzy fractal

2.1. Statement of the main theorem

Let $\mathbf{r}_0$ and $\mathbf{r}_1$ be the orthonormal basis vectors on the contracting plane, defined as follows:

$$\mathbf{r}_0 = (0.526, -0.747, -0.406), \quad \mathbf{r}_1 = (0, 0.478, -0.879) \quad (34)$$

The three-dimensional basis vectors $\mathbf{e}_0$, $\mathbf{e}_1$, and $\mathbf{e}_2$ are then projected onto the vectors $\mathbf{u}_0$, $\mathbf{u}_1$, and $\mathbf{u}_2$, respectively. These projected vector can be interpreted as two-dimensional vectors in terms of the orthonormal basis $\mathbf{r}_0$ and $\mathbf{r}_1$. The coordinates of these vectors are as follows:

$$\mathbf{u}_0 \approx (0.526, 0), \quad \mathbf{u}_1 \approx (-0.747, 0.478), \quad \mathbf{u}_2 \approx (-0.406, -0.879). \quad (35)$$

Let $[\mathbf{w}_n]$ be the sequence of words generated by the Rauzy substitution, as defined in Eqs. (3)–(7), and let $\mathbf{b}_n$ be the vector obtained by projecting the word vector $\mathbf{v}_n$ of the word $[\mathbf{w}_n]$, interpreted as two-dimensional vectors in terms of the orthonormal basis $\mathbf{r}_0$ and $\mathbf{r}_1$. Using the vectors defined in Eq. (35), the first five vectors of $\mathbf{b}_n$ can be expressed as follows:

$$\mathbf{b}_0 = \mathbf{u}_0, \quad (36)$$

$$\mathbf{b}_1 = \mathbf{u}_0 + \mathbf{u}_1, \quad (37)$$

$$\mathbf{b}_2 = 2\mathbf{u}_0 + \mathbf{u}_1 + \mathbf{u}_2, \quad (38)$$

$$\mathbf{b}_3 = 4\mathbf{u}_0 + 2\mathbf{u}_1 + \mathbf{u}_2, \quad (39)$$

$$\mathbf{b}_4 = 7\mathbf{u}_0 + 4\mathbf{u}_1 + 2\mathbf{u}_2. \quad (40)$$

For convenience, we introduce the following notation:

$$\mathbf{b}_n^{(i)} = \mathbf{b}_{3n+i}. \quad (41)$$

Let us state our first main theorem. We define the set A_n , referred to as the **A-layer of level n** , for $n \geq -1$, inductively as follows:

$$A_{-1} = \{\mathbf{0}\} \quad (42)$$

$$A_n = A_n^{(0)} \cup A_n^{(1)} \cup A_n^{(2)}, \quad \text{for } n \geq 0, \quad \text{where} \quad (43)$$

$$A_n^{(0)} = \{\mathbf{x} + \mathbf{b}_n^{(0)} \mid \mathbf{x} \in \bigcup_{i=-1}^{n-1} A_i\} \quad (44)$$

$$A_n^{(1)} = \{\mathbf{x} + \mathbf{b}_n^{(1)} \mid \mathbf{x} \in A_n^{(0)} \cup \bigcup_{i=-1}^{n-1} A_i\}, \quad (45)$$

$$A_n^{(2)} = \{\mathbf{x} + \mathbf{b}_n^{(2)} \mid \mathbf{x} \in A_n^{(0)} \cup A_n^{(1)} \cup \bigcup_{i=-1}^{n-1} A_i\}. \quad (46)$$

Theorem 1. The Rauzy fractal R is the union of all A-layers A_n .

$$R = \bigcup_{n \geq -1} A_n. \quad (47)$$

2.2. Lemmas for the proof of Theorem 1

Here we present the lemmas that will be used to prove Theorem 1. The first lemma is well-known:

Lemma 2. The word $[w_n]$ generated by the Rauzy substitution satisfies the following recursive formula:

$$[w_{n+3}] = [w_{n+2}][w_{n+1}][w_n], \quad n \geq 0. \quad (48)$$

Let us call the points in the Rauzy fractal R as **Rauzy point**. For example, the points b_n are Rauzy points for all $n \geq 0$. Recall that each Rauzy point is derived from a prefix of a word $[w_n]$. We define the **length** of a Rauzy point x is the length of its corresponding prefix and denote it by $L(x)$. Our second lemma is the following:

Lemma 3. If x is a Rauzy point that satisfies

$$L(x) \leq L(b_{n+1}) + L(b_n), \quad (49)$$

then $x + b_{n+2}$ is also a Rauzy point.

Proof. Let $[w]$ be the prefix of a Rauzy word that corresponds to the Rauzy point x . Then the word $[w]$ is a prefix of the concatenated word $[w_{n+1}][w_n]$ for some $n \geq 0$. Thus, by Lemma 2, the word $[w_{n+2}][w]$ is a prefix of $[w_{n+3}]$. Since $[w_{n+3}]$ contains $[w_{n+2}]$, we can conclude that $x + b_{n+2}$ is a Rauzy point.

The following is the corollary to Lemma 3.

Lemma 4. Let R_n denote the set of all Rauzy points whose lengths are less than or equal to n . Then, for all $n \geq 0$, the following holds:

$$A_n = R_{3n+3} - R_{3n} - \{b_{n+1}^{(0)}\}. \quad (50)$$

Proof. Let us prove by induction. The layer A_0 consists of the first six Rauzy points $x_0, \dots, x_5$, as illustrated in Fig. 7. With the point $x = 0$, these points are represented as follows:

$$x^{(0)} = b_0 \in A_0^{(0)} \quad (51)$$

$$x^{(1)} = b_1 \in A_0^{(1)} \quad (52)$$

$$x^{(2)} = x_0 + b_1 \in A_0^{(1)} \quad (53)$$

$$x^{(3)} = b_2 \in A_0^{(2)} \quad (54)$$

$$x^{(4)} = x_0 + b_2 \in A_0^{(2)} \quad (55)$$

$$x^{(5)} = x_1 + b_2 \in A_0^{(2)} \quad (56)$$

Assume that the identity (50) holds up to some index $n \geq 1$. According to the definition of A-layers in Eqs. (43)–(46), there are total 37 new Rauzy points in the subsequent layer A_{n+1} for each Rauzy point x in the A-layer of level $n-1$. Fig. 8 shows these points, together with the first six points that are in the A-layer of level n . These six points are represented as follows:

$$x_0 = x + b_n^{(0)} \in A_n^{(0)}, \quad (57)$$

$$x_1 = x + b_n^{(1)} \in A_n^{(1)}, \quad (58)$$

$$x_2 = x_0 + b_n^{(1)} \in A_n^{(1)}, \quad (59)$$

$$x_3 = x + b_n^{(2)} \in A_n^{(2)}, \quad (60)$$

$$x_4 = x_0 + b_n^{(2)} \in A_n^{(2)}, \quad (61)$$

$$x_5 = x_1 + b_n^{(2)} \in A_n^{(2)}. \quad (62)$$

Our goal is to show the 37 Rauzy points, ranging from x_6 to x_{42} , are indeed contained within A_{n+1} . Specifically, we will show that they belong to A_{n+1} in the following manner:

$$x_6, \dots, x_{12} \in A_n^{(0)}, \quad (63)$$

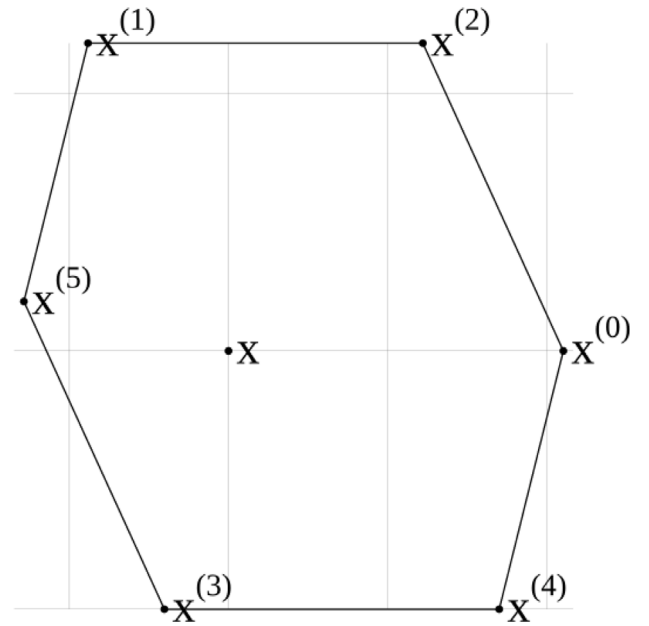


Fig. 7. The A-layers of the level 0 in the Rauzy fractal.

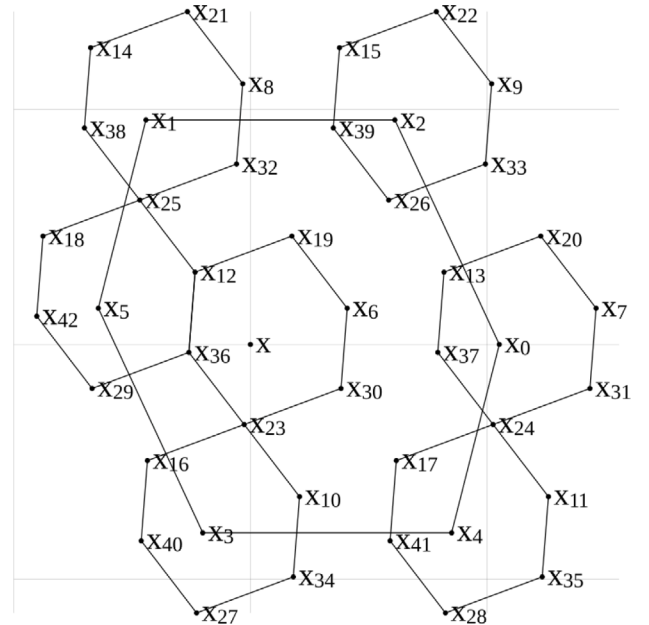


Fig. 8. The 43 Rauzy points in A-layers A_{n+1} generated by a point x in A_{n-1} .

$$x_{13}, \dots, x_{25} \in A_n^{(1)}, \quad (64)$$

$$x_{26}, \dots, x_{42} \in A_n^{(2)}. \quad (65)$$

First, let us prove the statement (63). We observe that

$$x_6 = x + b_n^{(0)}, \quad (66)$$

$$x_{i+7} = x_i + b_n^{(0)}, \quad \text{for } 0 \leq i \leq 5. \quad (67)$$

From Lemma 3, it is sufficient to show that, for $0 \leq i \leq 5$,

$$L(x), L(x_i) < L(b_{n-1}^{(2)}) + L(b_{n-1}^{(1)}). \quad (68)$$

From Eqs. (57)–(62), we obtain the following inequalities:

$$L(x) < L(b_{n-1}^{(0)}) < L(b_{n-1}^{(1)}), \quad (69)$$

$$L(\mathbf{x}_0) < L(\mathbf{b}_{n-1}^{(0)}) + L(\mathbf{b}_{n-1}^{(0)}) < L(\mathbf{b}_{n-1}^{(2)}), \quad (70)$$

$$L(\mathbf{x}_1) < L(\mathbf{b}_{n-1}^{(0)}) + L(\mathbf{b}_{n-1}^{(1)}) < L(\mathbf{b}_{n-1}^{(2)}), \quad (71)$$

$$L(\mathbf{x}_2) < L(\mathbf{b}_{n-1}^{(0)}) + L(\mathbf{b}_{n-1}^{(1)}) + L(\mathbf{b}_{n-1}^{(0)}) < L(\mathbf{b}_{n-1}^{(2)}) + L(\mathbf{b}_{n-1}^{(1)}), \quad (72)$$

$$L(\mathbf{x}_3) < L(\mathbf{b}_{n-1}^{(0)}) + L(\mathbf{b}_{n-1}^{(2)}) < L(\mathbf{b}_{n-1}^{(2)}) + L(\mathbf{b}_{n-1}^{(1)}), \quad (73)$$

$$L(\mathbf{x}_4) < L(\mathbf{b}_{n-1}^{(0)}) + L(\mathbf{b}_{n-1}^{(2)}) + L(\mathbf{b}_{n-1}^{(0)}) < L(\mathbf{b}_{n-1}^{(2)}) + L(\mathbf{b}_{n-1}^{(1)}), \quad (74)$$

$$L(\mathbf{x}_5) \leq L(\mathbf{b}_{n-1}^{(0)}) + L(\mathbf{b}_{n-1}^{(2)}) + L(\mathbf{b}_{n-1}^{(1)}). \quad (75)$$

The inequalities (69)–(74) prove the inequality (68) except for the case when $i = 5$. To cover this case, let us observe the representation of the point $\mathbf{x}_{12}$ in the following manner:

$$\mathbf{x}_{12} = \mathbf{x}_5 + \mathbf{b}_{n-1}^{(0)} = (\mathbf{x} + \mathbf{b}_{n-1}^{(1)} + \mathbf{b}_{n-1}^{(2)}) + \mathbf{b}_{n-1}^{(0)} = \mathbf{x} + \mathbf{b}_n^{(1)} \quad (76)$$

Thus, by Lemma 3, the point $\mathbf{x}_{12}$ is a Rauzy point. Since the length of the point $\mathbf{x}_{12}$ is the longest among points ranging from $\mathbf{x}_6$ to $\mathbf{x}_{12}$, and

$$\max L(\mathbf{x}_{12}) < L(\mathbf{b}_{n-1}^{(0)}) + L(\mathbf{b}_{n-1}^{(1)}) < L(\mathbf{b}_{n-1}^{(2)}), \quad (77)$$

the point $\mathbf{x}_{12}$ lies in $A_n^{(0)}$. This shows that $A_n^{(0)} \subset R_{n+1} - \{\mathbf{b}_{n+1}^{(1)}\}$.

Next, let us prove the statement (64). The points $\mathbf{x}_{13}, \dots, \mathbf{x}_{25}$ are represented as follows:

$$\mathbf{x}_{j+13} = \mathbf{x}_j + \mathbf{b}_n^{(1)}, \quad 0 \leq j \leq 12. \quad (78)$$

Thus the points from $\mathbf{x}_{13}$ to $\mathbf{x}_{22}$ satisfy the following inequality:

$$L(\mathbf{x}_i) \leq L(\mathbf{b}_{n-1}^{(2)}) + L(\mathbf{b}_n^{(0)}), \quad 0 \leq i \leq 9. \quad (79)$$

This implies that the points $\mathbf{x}_{13}, \dots, \mathbf{x}_{22}$ are Rauzy points. Meanwhile, the points $\mathbf{x}_{23}, \mathbf{x}_{24}$, and $\mathbf{x}_{25}$ satisfy the following inequalities:

$$\mathbf{x}_{23} = \mathbf{x}_3 + \mathbf{b}_n^{(0)} + \mathbf{b}_n^{(1)} = \mathbf{x} + \mathbf{b}_n^{(2)} + \mathbf{b}_n^{(0)} + \mathbf{b}_n^{(0)} = \mathbf{x} + \mathbf{b}_n^{(2)}, \quad (80)$$

$$\mathbf{x}_{24} = \mathbf{x}_4 + \mathbf{b}_n^{(0)} + \mathbf{b}_n^{(1)} = \mathbf{x}_0 + \mathbf{b}_{n-1}^{(2)} + \mathbf{b}_n^{(0)} + \mathbf{b}_n^{(1)} = \mathbf{x}_0 + \mathbf{b}_n^{(2)}, \quad (81)$$

$$\mathbf{x}_{25} = \mathbf{x}_5 + \mathbf{b}_n^{(0)} + \mathbf{b}_n^{(1)} = \mathbf{x}_1 + \mathbf{b}_{n-1}^{(2)} + \mathbf{b}_n^{(0)} = \mathbf{x}_1 + \mathbf{b}_n^{(2)}. \quad (82)$$

Therefore, Lemma 3 shows that $\mathbf{x}_{23}, \mathbf{x}_{24}, \mathbf{x}_{25}$ are Rauzy points. The point of the longest length, namely $\mathbf{x}_{25}$, satisfies the following inequality:

$$\max L(\mathbf{x}_{25}) < L(\mathbf{b}_n^{(0)}) + L(\mathbf{b}_n^{(2)}) < L(\mathbf{b}_{n+1}^{(0)}). \quad (83)$$

Therefore, we have $A_n^{(1)} \subset R_{n+1} - \{\mathbf{b}_{n+1}^{(0)}\}$.

Lastly, let us prove the statement (65). All points from $\mathbf{x}_{26}$ to $\mathbf{x}_{42}$ are obtained by adding $\mathbf{b}_n^{(2)}$ to the previously determined 26 points, ranging from $\mathbf{x}_0$ to $\mathbf{x}_{25}$.

$$\mathbf{x}_{j+24} = \mathbf{x}_j + \mathbf{b}_n^{(2)}, \quad 2 \leq j \leq 18. \quad (84)$$

The point of the longest length, namely the point $\mathbf{x}_{42}$, satisfies the following inequality:

$$L(\mathbf{x}_{42}) = L(\mathbf{x}_{18}) + L(\mathbf{b}_n^{(2)}) \quad (85)$$

$$= L(\mathbf{x}_5) + L(\mathbf{b}_n^{(1)}) + L(\mathbf{b}_n^{(2)}) \quad (86)$$

$$= L(\mathbf{x}) + L(\mathbf{b}_{n-1}^{(2)}) + L(\mathbf{b}_{n-1}^{(1)}) + L(\mathbf{b}_n^{(1)}) + L(\mathbf{b}_n^{(2)}) \quad (87)$$

$$< L(\mathbf{b}_{n-1}^{(0)}) + L(\mathbf{b}_{n-1}^{(2)}) + L(\mathbf{b}_{n-1}^{(1)}) + L(\mathbf{b}_n^{(1)}) + L(\mathbf{b}_n^{(2)}) \quad (88)$$

$$< L(\mathbf{b}_{n+1}^{(0)}). \quad (89)$$

Therefore, we have shown that $A_n^{(2)} \subset R_{3n+3} - \{\mathbf{b}_{n+1}^{(0)}\}$.

2.3. The proof of Theorem 1

From Lemma 4, we showed that

$$A_n^{(0)} \cup A_n^{(1)} \cup R_{3n+2} \subset (R_{3n+3} - \{\mathbf{b}_{n+1}^{(0)}\}) \quad (90)$$

This implies $A_n \subset R$ for all $n \geq 0$.

To prove the converse, suppose that $\mathbf{x}$ is a Rauzy point. Then one of the following inequalities on $\mathbf{x}$ must hold:

$$L(\mathbf{b}_n^{(0)}) \leq L(\mathbf{x}) < L(\mathbf{b}_n^{(1)}), \quad (91)$$

$$L(\mathbf{b}_n^{(1)}) \leq L(\mathbf{x}) < L(\mathbf{b}_n^{(2)}), \quad (92)$$

$$L(\mathbf{b}_n^{(2)}) \leq L(\mathbf{x}) < L(\mathbf{b}_{n+1}^{(0)}), \quad (93)$$

Let $[\mathbf{w}]$ be the prefix corresponding to the Rauzy point $\mathbf{x}$. If the inequality (91) true, then the word $[\mathbf{w}_n]$ is a prefix of $[\mathbf{w}]$, thus we can write $[\mathbf{w}]$ as

$$[\mathbf{w}] = [\mathbf{w}_n][\mathbf{w}']. \quad (94)$$

Since the word $[\mathbf{w}']$ is a prefix of $[\mathbf{w}_n]$, the point $\mathbf{x}'$ obtained by projecting the word vector of $[\mathbf{w}']$ onto the contracting plane is a Rauzy point, and is contained in A_{n-1} . Since $\mathbf{x} = \mathbf{x}' + \mathbf{b}_n^{(0)}$, we can conclude that $\mathbf{x} \in A_n$.

Next, suppose that the inequality (92) holds. Then $[\mathbf{w}]$, the prefix corresponding to the Rauzy point $\mathbf{x}$, is a prefix of $[\mathbf{w}_{3n+1}][\mathbf{w}']$. Note that the word $[\mathbf{w}']$ is a prefix of the concatenated word $[\mathbf{w}_{3n}][\mathbf{w}_{3n-1}]$. Since $\mathbf{x} = \mathbf{x}' + \mathbf{b}_n^{(1)}$, we have $\mathbf{x} \in A_n^{(1)}$. If $L(\mathbf{w}_{3n}) \leq L([\mathbf{w}'])$, then we can write the word $[\mathbf{w}']$, the word corresponding to the point $\mathbf{x}'$, as follows:

$$[\mathbf{w}'] = [\mathbf{w}_{3n}][\mathbf{w}'']. \quad (95)$$

Here, $[\mathbf{w}']$ is again a prefix of $[\mathbf{w}_{3n-1}]$. Let $\mathbf{x}''$ be the projection of the word vector of $[\mathbf{w}'']$ onto the contracting point. Since it satisfies $\mathbf{x} = \mathbf{x}'' + \mathbf{b}_n^{(0)} + \mathbf{b}_n^{(1)}$, the point $\mathbf{x}''$ is a Rauzy point in $A_n^{(1)}$, and this proves that $\mathbf{x} \in A_n$.

Finally, if the inequality (93) holds, then the word $[\mathbf{w}]$, again the prefix corresponding to the Rauzy point $\mathbf{x}$, is a prefix of $[\mathbf{w}_{3n+2}][\mathbf{w}']$, and the word $[\mathbf{w}']$ is also a prefix of $[\mathbf{w}_{3n+1}][\mathbf{w}_{3n}]$. By the similar argument above, we have $\mathbf{x} \in A_n^{(2)}$, thus prove that $\mathbf{x} \in A_n$.

2.4. The construction of the boundary of the Rauzy fractal

Here we present a method to construct the boundary of the Rauzy fractal using the A-layered structure. We introduce some terminology related to the ancestry of Rauzy points. For a Rauzy point $\mathbf{x}$ in the A-layer A_n of the level n , the six points defined in Eqs. (57)–(62) are called the **children** of $\mathbf{x}$. We call the point $\mathbf{x}$ the **parent** of these children. For example, the six points from $\mathbf{x}_0$ to $\mathbf{x}_5$ in Fig. 7 are the children of the point $\mathbf{0}$ in the A-layer A_{-1} . The terms **grandchildren** and **grandparent** are defined similarly, referring to the second and previous generations in the hierarchical structure of Rauzy points.

For each Rauzy point $\mathbf{x}$, the six children of $\mathbf{x}$ form a convex hexagonal region. This region is referred to as the **house** of $\mathbf{x}$. Fig. 8 visualizes the houses of the Rauzy points from $\mathbf{0}$ (labeled as $\mathbf{x}$ in Fig. 8), $\mathbf{x}_0, \dots, \mathbf{x}_5$. Note that a point may have more than one parents; the points $\mathbf{x}_{12}, \mathbf{x}_{23}, \mathbf{x}_{24}, \mathbf{x}_{25}$, and $\mathbf{x}_{36}$ have two parents each.

The plot the boundary of the Rauzy fractal effectively, we can discard the points that lie in the houses of their grandparents. For example, in Fig. 8, we can observe that the fourteen grandchildren of the points $\mathbf{x}$, namely $\mathbf{x}_6, \mathbf{x}_{10}, \mathbf{x}_{12}, \mathbf{x}_{13}, \mathbf{x}_{17}, \mathbf{x}_{19}, \mathbf{x}_{23}, \mathbf{x}_{25}, \mathbf{x}_{26}, \mathbf{x}_{30}, \mathbf{x}_{32}, \mathbf{x}_{36}, \mathbf{x}_{37}$, and $\mathbf{x}_{39}$, are located within the house of their grandparent. These points can be considered interior points of the Rauzy fractal and, therefore, can be excluded from further offspring generation.

To plot the boundary of the Rauzy fractal, we follow these steps recursively, beginning with the six Rauzy points in A_0 .

1. For each Rauzy point $\mathbf{x}$ in the A-layer A_n , for $n \geq 0$, generate its children.
2. If the children of $\mathbf{x}$ lies within the house of grandparent, or one of grandparents, of $\mathbf{x}$, discard them.
3. Repeat step 1 with the remaining children.

The children of the Rauzy points that are not excluded from generating their offspring are referred to as the **boundary points**. Figs. 9 and 10 illustrate the boundary points up to levels 2 and 3, respectively, together with the houses of their ancestors.

Fig. 11 displays the boundary points up to level 4, without showing the houses of their ancestors. A notable feature of this figure is that the boundary points tend to penetrate the interior of the Rauzy fractal. This tendency is already evident in previous figures illustrating the

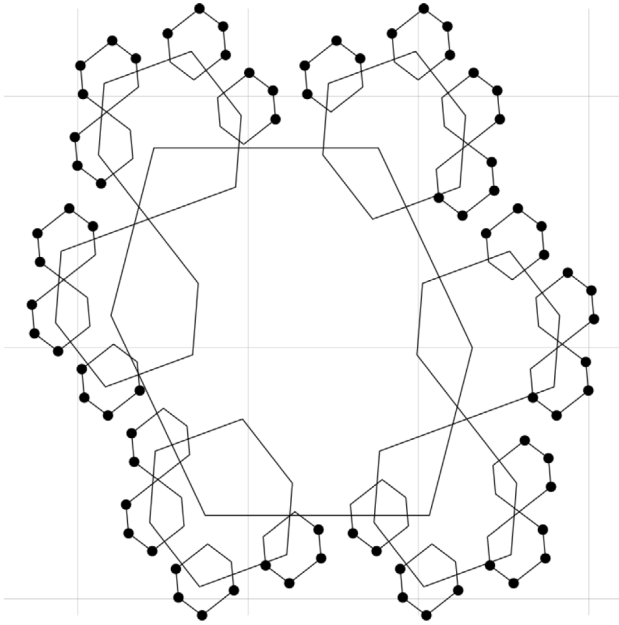


Fig. 9. The boundary points in A-layers up to level 2.

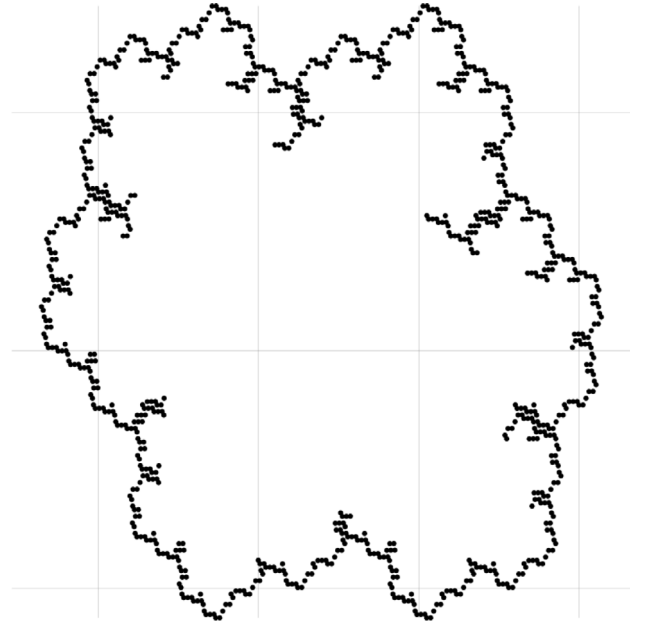


Fig. 11. The boundary points in A-layers up to the level 4.

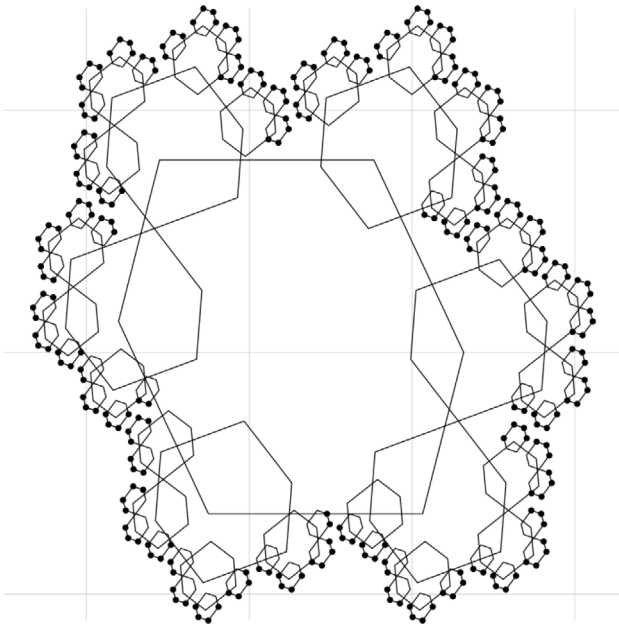


Fig. 10. The boundary points in A-layers up to level 3.

boundary points. The phenomenon arises due to the limited precision of the Rauzy fractal's boundary when visualized within the A-layer of a given level. An attempt was made to eliminate cases where boundary points appear to lie inside the Rauzy fractal by selecting only the boundary points that define the convex hull. However, this approach resulted in a loss of accuracy in representing the boundary of the Rauzy fractal. Once the gaps between the boundary points are removed at a certain level, the precision of the boundary cannot be restored, as subsequent boundary points are generated based on the modified, less precise set of points. Therefore, retaining the boundary points in their original positions, even when they appear to be located within

Table 2

Rauzy points at each A-layers.

level	$ A_n $	Number of boundary points in A_n
0	6	6
1	44	22
2	274	77
3	1705	266
4	10 609	915
5	66 012	3144

the interior of the Rauzy fractal, is the most reliable approach to maintaining the precision of the fractal's boundary.

Nevertheless, we confirmed that our method can significantly reduce the computational cost of plotting the boundary of the Rauzy fractal. Table 2 presents the number of points in the A-layer A_n for $n \leq 5$ and the corresponding number of boundary points selected at each level. Although the number of boundary points increases exponentially with the level, the total number of points to be plotted is substantially reduced compared to the total number of points in the A-layer.

3. B-layered structure of the Rauzy fractal

3.1. Statement of the main theorem

We present another method of constructing the Rauzy fractal, relying on the fact that the Rauzy fractal is the union of the B-layers. Let $\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2$ be the two-dimensional vectors defined in Eqs. (35). Then we define the vectors $\mathbf{s}_0^{(i)}$, for $i = 0, \dots, 6$, as follows

$$\mathbf{s}_0^{(0)} = \mathbf{s}_0^{(2)} = \mathbf{s}_0^{(4)} = \mathbf{s}_0^{(6)} = \mathbf{u}_0, \quad (96)$$

$$\mathbf{s}_0^{(1)} = \mathbf{s}_0^{(5)} = \mathbf{u}_1, \quad (97)$$

$$\mathbf{s}_0^{(3)} = \mathbf{u}_2. \quad (98)$$

For each $n \geq 1$ and $i = 0, \dots, 6$, define vectors $\mathbf{s}_n^{(i)}$ as follows.

$$\mathbf{s}_{n+1}^{(0)} = \mathbf{s}_n^{(0)} + \mathbf{s}_n^{(1)} + \mathbf{s}_n^{(2)} + \mathbf{s}_n^{(3)} + \mathbf{s}_n^{(4)} + \mathbf{s}_n^{(5)} + \mathbf{s}_n^{(6)}, \quad (99)$$

$$\mathbf{s}_{n+1}^{(1)} = \mathbf{s}_n^{(0)} + \mathbf{s}_n^{(1)} + \mathbf{s}_n^{(2)} + \mathbf{s}_n^{(3)} + \mathbf{s}_n^{(4)} + \mathbf{s}_n^{(5)}, \quad (100)$$

$$\mathbf{s}_{n+1}^{(2)} = \mathbf{s}_n^{(0)} + \mathbf{s}_n^{(1)} + \mathbf{s}_n^{(2)} + \mathbf{s}_n^{(3)} + \mathbf{s}_n^{(4)} + \mathbf{s}_n^{(5)} + \mathbf{s}_n^{(6)}, \quad (101)$$

$$\mathbf{s}_{n+1}^{(3)} = \mathbf{s}_n^{(0)} + \mathbf{s}_n^{(1)} + \mathbf{s}_n^{(2)} + \mathbf{s}_n^{(3)}, \quad (102)$$

$$s_{n+1}^{(4)} = s_n^{(0)} + s_n^{(1)} + s_n^{(2)} + s_n^{(3)} + s_n^{(4)} + s_n^{(5)} + s_n^{(6)}, \quad (103)$$

$$s_{n+1}^{(5)} = s_n^{(0)} + s_n^{(1)} + s_n^{(2)} + s_n^{(3)} + s_n^{(4)} + s_n^{(5)}, \quad (104)$$

$$s_{n+1}^{(6)} = s_n^{(0)} + s_n^{(1)} + s_n^{(2)} + s_n^{(3)} + s_n^{(4)} + s_n^{(5)} + s_n^{(6)}. \quad (105)$$

Finally, let B_n , for $n \geq -1$, be the set defined recursively as follows:

$$B_{-1} = \{0\} \quad (106)$$

$$B_n = \{x + \sum_{i=0}^6 s_n^{(i)} \mid x \in \bigcup_{i=-1}^{n-1} B_i\}. \quad (107)$$

We will call the set B_n as the **B-layer of the level n** . Our main theorem is as follows.

Theorem 5. The Rauzy fractal R is the union of all B-layers B_n

$$R = \bigcup_{n \geq -1} B_n \quad (108)$$

3.2. The proof of Theorem 5

We will show that the following holds for all $n \geq 0$:

$$A_n \subset B_n \subset A_{n+1}^{(0)}. \quad (109)$$

This follows immediately from the following equalities:

$$s_n^{(0)} = s_n^{(2)} = s_n^{(4)} = s_n^{(6)} = b_n^{(0)}, \quad (110)$$

$$s_n^{(1)} = s_n^{(5)} = b_n^{(1)} - b_n^{(0)}, \quad (111)$$

$$s_n^{(3)} = b_n^{(2)} - b_n^{(0)} - b_n^{(1)}, \text{ for } n \geq 0. \quad (112)$$

We will prove these equations by induction. From the definition of $s_0^{(i)}$ for $0 \leq i \leq 6$ in Eqs. (96)–(98), we have the following identities:

$$s_0^{(0)} = s_0^{(2)} = s_0^{(4)} = s_0^{(6)} = u_0 = b_0^{(0)}, \quad (113)$$

$$s_0^{(1)} = s_0^{(5)} = u_1 = b_0^{(1)} - b_0^{(0)}, \quad (114)$$

$$s_0^{(3)} = u_2 = b_0^{(2)} - b_0^{(0)} - b_0^{(1)}. \quad (115)$$

Suppose that Eqs. (110)–(112) hold up to $n \geq 0$. Then we have the following equalities for $n+1$:

$$\begin{aligned} s_{n+1}^{(0)} &= s_{n+1}^{(2)} = s_{n+1}^{(4)} = s_{n+1}^{(6)} = b_n^{(0)} + b_n^{(1)} + b_n^{(2)} = b_{n+1}^{(0)}, \\ s_{n+1}^{(1)} &= s_{n+1}^{(5)} = b_n^{(1)} + b_n^{(2)} = b_{n+1}^{(1)} - b_{n+1}^{(0)}, \\ s_{n+1}^{(3)} &= b_n^{(2)} = b_{n+1}^{(2)} - b_{n+1}^{(0)} - b_{n+1}^{(1)}. \end{aligned} \quad (116)$$

As a consequence, we have the following equalities for $n \geq 0$:

$$s_n^{(0)} = b_n^{(0)}, \quad (117)$$

$$s_n^{(0)} + s_n^{(1)} = b_n^{(1)}, \quad (118)$$

$$s_n^{(0)} + s_n^{(1)} + s_n^{(2)} = b_n^{(0)} + b_n^{(1)}, \quad (119)$$

$$s_n^{(0)} + s_n^{(1)} + s_n^{(2)} + s_n^{(3)} = b_n^{(2)}, \quad (120)$$

$$s_n^{(0)} + s_n^{(1)} + s_n^{(2)} + s_n^{(3)} + s_n^{(4)} = b_n^{(0)} + b_n^{(2)}, \quad (121)$$

$$s_n^{(0)} + s_n^{(1)} + s_n^{(2)} + s_n^{(3)} + s_n^{(4)} + s_n^{(5)} = b_n^{(1)} + b_n^{(2)}, \quad (122)$$

$$s_n^{(0)} + s_n^{(1)} + s_n^{(2)} + s_n^{(3)} + s_n^{(4)} + s_n^{(5)} + s_n^{(6)} = b_n^{(0)} + b_n^{(1)} + b_n^{(2)} = b_{n+1}^{(0)}. \quad (123)$$

This proves the theorem.

3.3. The construction of the boundary of the Rauzy fractal using the B-layers

From the definition of $s_n^{(i)}$ for $0 \leq i \leq 6$ in Eqs. (99)–(105), it is evident that we are using the pseudo self-replicating pattern of word [0102010] introduced in Section 1.4. The vector $s_n^{(i)}$ is the projected point of the word vector corresponding to the i th token of the n th replicate of the word [0102010].

Fig. 12 shows the Rauzy points in the B-layers up to the level 1. The terminology introduced in Section 2.4, such as the parent, children,

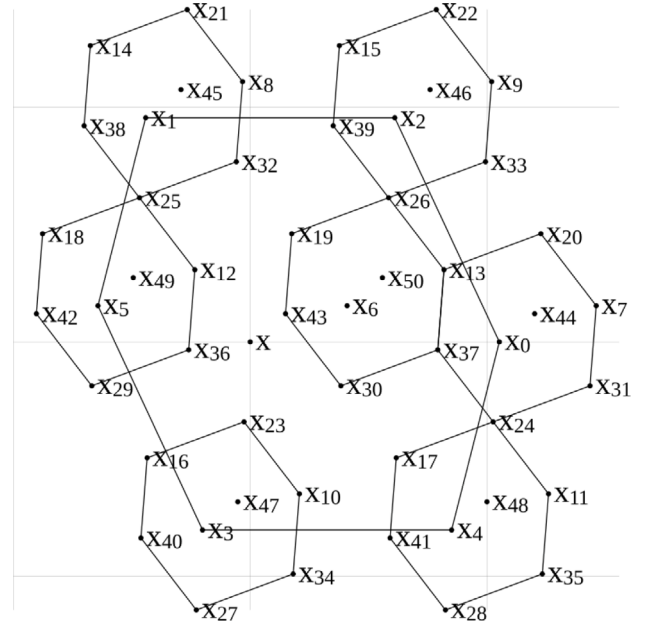


Fig. 12. Rauzy points in the B-layers up to the level 2.

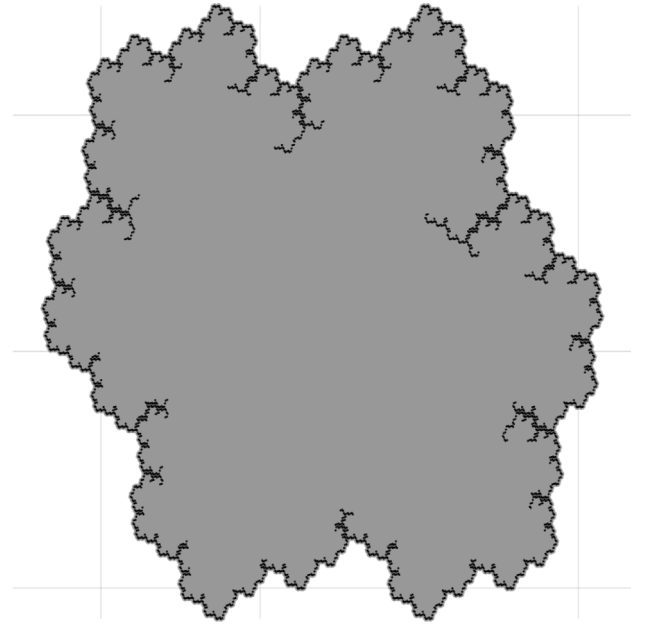


Fig. 13. The boundary of the Rauzy fractal constructed by B-layers.

grandchildren, and grandparent, can be applied to the B-layers as well. The figure illustrates the houses of the Rauzy points 0 (labeled as x), $x_0, \dots, x_6$ in the B-layers. Since each point in the B-layer B_n generates seven, rather than six, children, there are slightly more Rauzy points in the B-layers B_1 than in the A-layers A_1 .

The same method for constructing the boundary of the Rauzy fractal can also be applied to the B-layers. Fig. 13 illustrates the boundary of the Rauzy fractal constructed using the B-layers, where the boundary points are highlighted in red. The tendency of boundary points to penetrate the interior of the Rauzy fractal still remains. As discussed in Section 2.4, this phenomenon arises due to the limited precision of the Rauzy fractal's boundary when visualized within the B-layer at a given level.

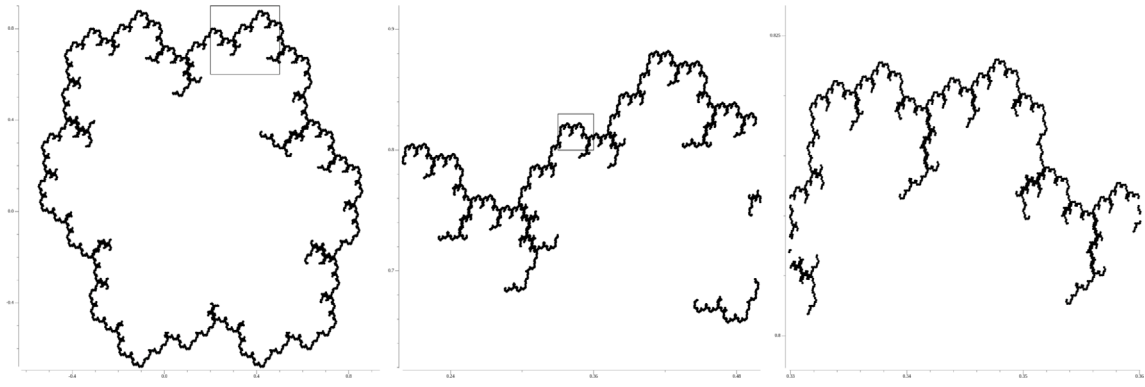


Fig. 14. The magnification of the boundary of the Rauzy fractal constructed by B-layers.

An important advantage of constructing the boundary of the Rauzy fractal using the A- and B-layered structures is the ability to magnify any arbitrary region of the fractal. This approach is particularly useful for two reasons: first, the fractal-like behavior of the Rauzy fractal is exhibited primarily on its boundary; Second, it eliminates the need to plot unnecessary interior points, allowing the magnification of specific regions with the computational efficiency.

Fig. 14 presents a series of magnified images of the boundary of the Rauzy fractal, constructed using the B-layers. The first image is plotted within the region $[-0.55, 0.86] \times [-0.68, 0.88]$, using B-layers up to level 6. The second image is a magnification of the region $[0.2, 0.5] \times [0.6, 0.9]$ from the first image. Simply magnifying the first image would not achieve the same level of precision, as the visualization of the boundary is based on the discrete plotting of points. A direct magnification would merely widen the gaps between boundary points. Instead, the second image was obtained by selecting boundary points in the first image within the specified region, discarding all other points outside the region, and generating offspring only within it. While the first image contains all boundary points in the B-layer of level 6, the second image includes only the boundary points in the B-layer of level 8 within this region. Plotting the entire boundary points of the Rauzy fractal in the B-layer up to 8 requires a significant amount of computational resources, but by selecting only the boundary points within the specified region, we can achieve the same level of detail with a fraction of the computational cost. The third image is a further magnification of the region $[0.33, 0.36] \times [0.80, 0.83]$ from the second image. Similarly, boundary points in the B-layer of level 8 within this region were selected, and offspring were generated up to the B-layer at level 10. By following this approach, we achieve a consistent level of detail in the magnified boundary of the Rauzy fractal, accurately capturing finer structures without the need to plot unnecessary interior points.

4. Self-replicating words and discrete tilings

4.1. Definition of self-replicating words

Here we introduce a generalization of the pseudo self-replicating pattern of the word $[0102010]$. Let us restrict the choice of letters to either 0, 1, or 2. The **self-replicating pattern** of the word $[w_0]$ is the sequence of words $[w_n]$ generated by the following rule:

- Let $[w_0] = [w_0 \dots w_l]$ be the initial word.
- Each word $[w_n]$, for $n \geq 0$, is the concatenation of the words $[w_n^{(0)}], \dots, [w_n^{(l)}]$. The initial word $[w_0]$ is tokenized by single-lettered words: $[w_0^{(i)}] = [w_i]$ for $i = 0, \dots, l$.



Fig. 15. The domain produced by self-replicating pattern of the word $[0102]$.

- The i th token of the word $[w_{n+1}]$ is words $[w_n]$ after trimming the last i tokens of $[w_n]$.

The word $[w_n]$ is called the n th **replicate** of the word $[w_0]$.

4.2. Examples of self-replicating words

Let us consider the self-replicating pattern of the word $[w_0] = [0120]$. The first three replicates of this pattern are the following:

$$[w_0] = [0120] \quad (124)$$

$$[w_1] = \underbrace{[0120]}_{0 \text{ trim}} \mid \underbrace{[012]}_{1 \text{ trim}} \mid \underbrace{[01]}_{2 \text{ trims}} \mid \underbrace{[0120]}_{0 \text{ trim}} \quad (125)$$

$$[w_2] = \underbrace{[0120012010120]}_{0 \text{ trim}} \mid \underbrace{[012001201]}_{1 \text{ trim}} \mid \underbrace{[0120012]}_{2 \text{ trims}} \mid \underbrace{[0120012010120]}_{0 \text{ trim}} \quad (126)$$

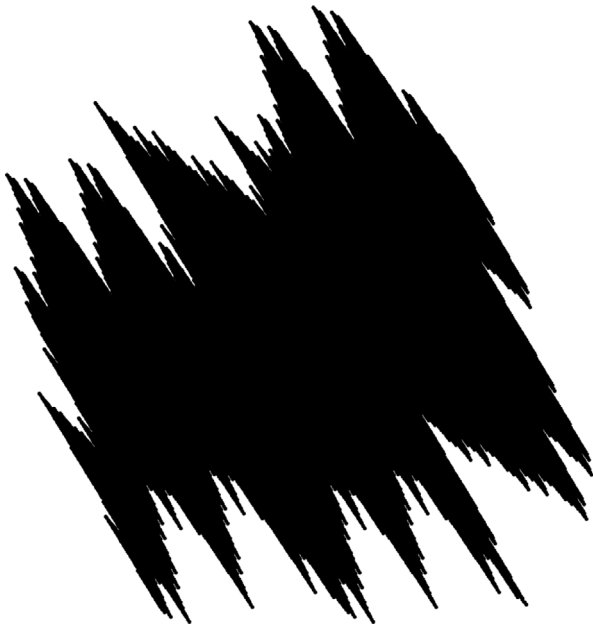


Fig. 16. The domain produced by self-replicating pattern of the word [0201].

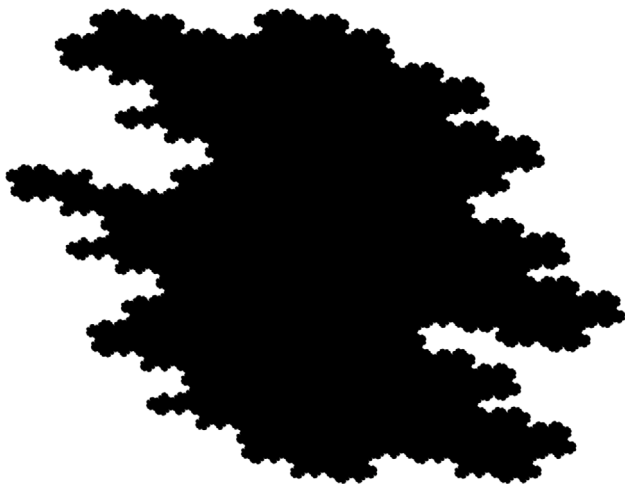


Fig. 17. The domain produced by self-replicating pattern of the word [0102010].

We can plot the domain for such self-replicating words in a similar way to the Rauzy fractal. For sufficiently large n , we associate the word vector to prefixes of the replicate $[w_n]$. The contracting plane is identified by the hyperplane orthogonal to the asymptotic direction of these word vector. Then we can project the word vectors to the contracting plane. We discovered that these points seem to cluster into a compact region for special choice of the initial word $[w_0]$. For example, self-replicating pattern of the word $[w_0] = [0120]$ can produce the compact domain shown in Fig. 5.

Other similar examples of self-replicating words are also found. Figs. 15–19 show the domains of six self-replicating words. The initial words are described in the caption of each figure.

One notable feature of these domains is that they can tile two-dimensional plane discretely. Figs. 6 and 20 shows the tiling of the plane by the self-replicating pattern of the word [0120] and [2010], respectively. Such self-replicating patterns are implemented in a code, which can be found at <https://github.com/HyosangKang/srw>



Fig. 18. The domain produced by self-replicating pattern of the word [1201].

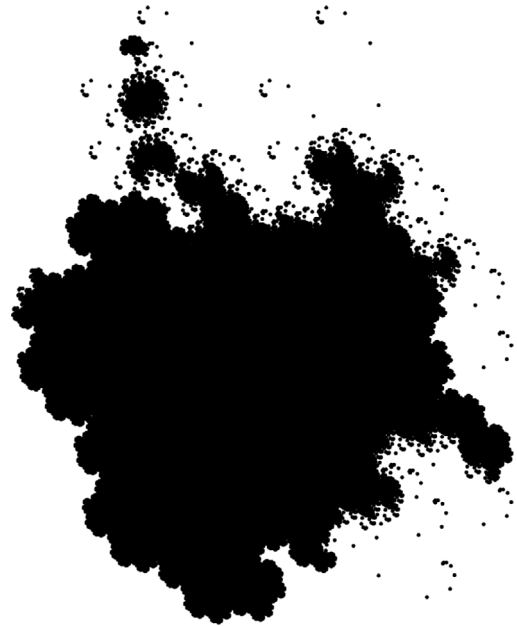


Fig. 19. The domain produced by self-replicating pattern of the word [2010].

However, the self-replicating patterns of words such as [0120] and [1201] do not always result in a compact domain that tiles the two-dimensional plane. For instance, the self-replicating pattern of the word [0] generates a sequence of words of increasing length, all consisting of the letter 0. Although the word vectors associated with these words converge asymptotically, their projections onto the contracting plane collapse to a single point. It would be an interesting direction for future research to investigate the conditions under which the self-replicating pattern of a word produces a compact domain that can tile the two-dimensional plane.

5. Conclusion

We utilized the characteristics of the Rauzy fractal to construct its boundary, demonstrating that it can be constructed effectively. Furthermore, our discovery of self-replicating words in relation to the

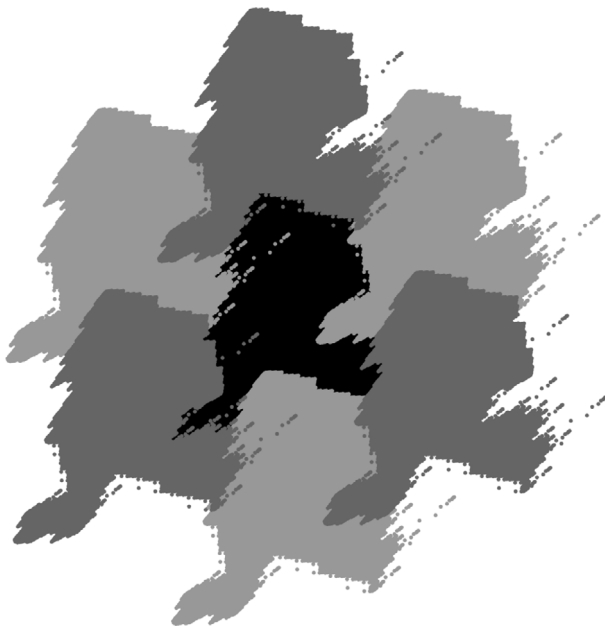


Fig. 20. The discrete tiling of the compact domain for the self-replicating pattern of [2010].

Rauzy fractal suggests a broader generalization to the compactness of domains formed by self-replicating words and their tiling properties. Extending the concept of self-replicating words to higher dimensions presents an intriguing future research. Additionally, developing more efficient methods to enhance the resolution of the fractal's boundary could be another promising direction for further study.

CRedit authorship contribution statement

Hyosang Kang: Writing – review & editing, Writing – original draft, Validation, Investigation, Data curation, Conceptualization. **Woojin Choi:** Validation, Investigation. **Jeonghoon Rhee:** Validation, Investigation. **Youchan Oh:** Validation, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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